SCATTERING THEORY OF DISCRETE (PSEUDO) LAPLACIANS ON A WEYL CHAMBER

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Abstract. To a crystallographic root system we associate a system of multivariate orthogonal polynomials diagonalizing an integrable system of discrete pseudo Laplacians on the Weyl chamber. We develop the time-dependent scattering theory for these discrete pseudo Laplacians and determine the corresponding wave operators and scattering operators in closed form. As an application, we describe the scattering behavior of certain hyperbolic Ruijsenaars-Schneider type lattice Calogero-Moser models associated with the Macdonald polynomials.

Contents

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1. Introduction

A fundamental property of the solitonic solutions of integrable nonlinear wave equations is that their multi-particle scattering process decomposes into pairwise two-particle interactions [SCM, AS, N-Z, N, FT]. This phenomenon is preserved at the quantum level: the corresponding solitonic quantum field theories are characterized by an N-particle scattering matrix that factorizes in terms of two-particle scattering matrices [M, KBI]. As it turns out, this type of factorization can be understood heuristically as being a consequence of the integrability of the models in question [Ku, RSc].

An archetype example of an integrable system with factorized scattering is the celebrated nonlinear Schrödinger equation (NLS). The quantum version of this model boils down to a bosonic N-particle system with a pairwise interaction via delta-functional potentials. The factorization of the scattering manifests itself through the asymptotics of the wave function, which is characterized by (products of) two-particle scattering matrices (or c-functions) [M, Ga, Ox, KBI].

In recent work, Ruijsenaars constructed a remarkably large class of quantum integrable lattice models of N-particles exhibiting factorized scattering [R4]. The discrete systems in question arise by interpreting recurrence relations (or Pieri formulas) for symmetric multivariate orthogonal polynomials as quantum eigenvalue equations. Here the polynomial variable plays the role of the spectral parameter and the index (i.e. partition) labelling the polynomials is thought of as the discrete spatial variable. By analyzing the asymptotics of the polynomials as the degree tends to infinity, Ruijsenaars demonstrated that—for factorized orthogonality measures subject to certain technical conditions ensuring that the particle interaction is short-range and the spectrum is absolutely continuous—the corresponding discrete models are governed by a scattering matrix that factorizes into two-particle scattering matrices. An interesting particular case is that of the Macdonald polynomials [M2]. The corresponding N-particle model can be identified as a hyperbolic Ruijsenaars-Schneider type lattice Calogero-Moser system [R1, R3]. At the level of classical Hamiltonian mechanics, the scattering of the corresponding integrable system was studied in great detail in Ref. [R2].

It is known that the root systems of simple Lie algebras form a fruitful context for understanding Calogero-Moser systems and particle models with delta-functional potentials [OP, Gu, HS, HO, O]. From this perspective, it is natural to ask for a generalization of Ruijsenaars' construction to the case of arbitrary root systems. The purpose of the present paper is to provide such a construction.

More specifically, we associate to a crystallographic root system a system of Weyl-group invariant multivariate orthogonal polynomials on the Weyl alcove, characterized by a weight function that factorizes over the roots (of the root system) in terms of one-dimensional c-functions. The orthogonality implies that the polynomials satisfy a system of recurrence relations (Pieri formulas). These recurrence relations are interpreted as eigenvalue equations for an integrable system of discrete pseudo Laplacians on the Weyl chamber. We develop the time-dependent scattering theory for these discrete pseudo Laplacians and determine the corresponding

wave operators and scattering operators in closed form. For a specific choice of the weight function, our polynomials amount to the Macdonald polynomials associated with root systems [M3, M4]. Again the corresponding integrable lattice model then permits identification as a discrete hyperbolic Ruijsenaars-Schneider type Calogero-Moser system [R1, R3, D1]. For the type A root systems the Weyl group is the symmetric group and we reproduce the results of Ruijsenaars [R4].

The wave- and scattering operators computed in this paper compare the dynamics generated by the discrete pseudo Laplacian to that of a free discrete Laplacian (corresponding to the case that the c-functions reduce to constant functions). Our study of the scattering consists of two parts. In the first (time-independent) part it is shown that the wave function of the discrete pseudo Laplacian has plane wave asymptotics, provided that the c-functions determining the orthogonality measure of the polynomials satisy certain analyticity requirements (guaranteeing that the spectrum of the discrete pseudo Laplacian is absolutely continuous). This part of the discussion hinges on previous results describing the large-degree asymptotics of the class of multivariate orthogonal polynomials under consideration [D3, D4]. The second (time-dependent) part consists of a stationary phase analysis that permits proving the existence and unitarity of the wave operators and scattering operators given the plane wave asymptotics of the wave functions. Key ingredient of this part of the discussion is a stationary phase estimate from [RS, p. 38-39] that controls the decay for $t \to \pm\infty$ of certain oscillatory integrals describing the difference between interacting and freely evolving wave packets.

The paper is organized as follows. Section 2 describes the construction of orthogonal polynomials related to root systems. In Section 3 we introduce a commuting system of discrete pseudo Laplacians on the Weyl chamber diagonalized by the orthogonal polynomials in question. The wave operators and scattering operators for our discrete pseudo Laplacians are determined in Section 4. The stationary phase analysis that lies at the basis of the computation of these wave– and scattering operators is relegated to Section 5. Finally, in Section 6 we specialize to the case of Macdonald polynomials and detail the scattering theory of the associated hyperbolic Ruijsenaars-Schneider type lattice Calogero-Moser models. Some key properties of the Macdonald polynomials invoked in Section 6 have been collected in Appendix A at the end of the paper. For the reader's convenience, we have also included an index of notations in Appendix B.

Let us conclude this introduction by providing a brief description of what the main results amount to in the elementary (classical) situation of a root system of rank 1. Let $\hat{c}(z)$ be a zero-free analytic function on the disc $|z| \leq \varrho$, with $\varrho > 1$, that is real-valued for z real and normalized such that $\hat{c}(0) = 1$. We associate to $\hat{c}(z)$ an orthonormal basis of trigonometric polynomials $P_0(\xi), P_1(\xi), P_2(\xi), \ldots$ for the Hilbert space $L^2((0,\pi), \frac{2\sin^2(\xi)\mathrm{d}x}{\pi\hat{c}(e^{i\xi})\hat{c}(e^{-i\xi})})$ that is obtained by applying the Gram-Schmidt process to the Fourier-cosine basis $1, \cos(\xi), \cos(2\xi), \ldots$. It is an immediate consequence of the three-term recurrence relation for the orthonormal polynomials $P_\ell(\xi)$ that the wave function

$$\Psi_\ell(\xi) = \frac{2\sin(\xi)P_\ell(\xi)}{\sqrt{\hat{c}(e^{i\xi})\hat{c}(e^{-i\xi})}}, \quad \xi \in (0,\pi),\ \ell \in \mathbb{N}, \tag{1.1}$$

satisfies an eigenvalue equation of the form $L\Psi = 2\cos(\xi)\Psi$, where L represents a discrete (self-adjoint) Laplacian acting on lattice functions $\phi \in \ell^2(\mathbb{N})$ as

$$L\phi_\ell = a_\ell \phi_{\ell+1} + b_\ell \phi_\ell + a_{\ell-1}\phi_{\ell-1} \qquad (\phi_{-1} \equiv 0), \tag{1.2}$$

with a_ℓ, b_ℓ denoting the coefficients of the three-term recurrence relation. For $\hat{c}(z) = 1$, the polynomials $P_\ell(\xi)$ amount to the Chebyshev polynomials of the second kind $U_\ell(\cos\xi) = \sin(\ell+1)\xi/\sin\xi$, whence the wave function in Eq. (1.1) reduces in this case to the Fourier-sine kernel $\Psi_\ell^{(0)}(\xi) = 2\sin(\ell+1)\xi$. The Laplacian L (1.2) then amounts to a free Laplacian $L^{(0)}$ whose action on lattice functions is given by $L^{(0)}\phi_\ell = \phi_{\ell+1} + \phi_{\ell-1}$.

Theorem 4.1 (below) now states that for $\ell \to \infty$ the wave function $\Psi_\ell(\xi)$ (1.1) converges exponentially fast in $L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ to the anti-symmetric combination of plane waves

$$\Psi_l^\infty(\xi) = \hat{s}^{1/2}(\xi)e^{i(\ell+1)\xi} - \hat{s}^{-1/2}(\xi)e^{-i(\ell+1)\xi}, \tag{1.3}$$

with $\hat{s}(\xi) = \hat{c}(e^{-i\xi})/\hat{c}(e^{i\xi})$.

Furthermore, let us denote by $\mathcal{F} : l^2(\mathbb{N}) \mapsto L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ and $\mathcal{F}^{(0)} : l^2(\mathbb{N}) \mapsto L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ the Fourier pairings with kernel $\Psi_\ell(\xi)$ and $\Psi_\ell^{(0)}(\xi)$, respectively:

$$\begin{cases} \hat{\phi}(\xi) = \sum_{\ell\in\mathbb{N}} \phi_\ell \Psi_\ell(\xi) \\ \phi_\ell = \frac{1}{2\pi}\int_0^\pi \hat{\phi}(\xi)\Psi_\ell(\xi)\mathrm{d}\xi \end{cases}, \qquad \begin{cases} \hat{\phi}(\xi) = \sum_{\ell\in\mathbb{N}} \phi_\ell \Psi_\ell^{(0)}(\xi) \\ \phi_\ell = \frac{1}{2\pi}\int_0^\pi \hat{\phi}(\xi)\Psi_\ell^{(0)}(\xi)\mathrm{d}\xi \end{cases}. \tag{1.4}$$

Then Theorem 4.2 and Corollary 4.3 (below) state that the wave operators $\Omega_\pm = s-\lim_{t\to\pm\infty} e^{itL}e^{-itL^{(0)}}$ and the scattering operator $\mathcal{S} = \Omega_+^{-1}\Omega_-$ exist in $l^2(\mathbb{N})$ and are given explicitly by the unitary operators $\Omega_\pm = \mathcal{F}^{-1}\circ\hat{\mathcal{S}}^{\mp 1/2}\circ\mathcal{F}^{(0)}$ and $\mathcal{S} = (\mathcal{F}^{(0)})^{-1}\circ\hat{\mathcal{S}}\circ\mathcal{F}^{(0)}$, where $\hat{\mathcal{S}}$ denotes a unitary scattering matrix that is characterized by its multiplicative action on a wave packet $\hat{\phi} \in L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ of the form $(\hat{\mathcal{S}}\hat{\phi})(\xi) = \hat{s}(-\xi)\hat{\phi}(\xi)$ for $0 < \xi < \pi$ (with $\hat{s}(\xi)$ as defined just below Eq. (1.3)).

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2. Orthogonal Polynomials Related to Root Systems

In this section we introduce a class of multivariate orthogonal polynomials related to root systems. For basic facts on root systems we refer to the standard works [B, Hu].

2.1. **Polynomials on the Weyl Alcove.** Let $\mathbf{E}$, $\langle\cdot,\cdot\rangle$ be a real N-dimensional Euclidean vector space and let $\boldsymbol{R} \subset \mathbf{E}$ denote an irreducible crystallographic root system spanning $\mathbf{E}$. We write $\mathcal{Q}$ and $\mathcal{Q}^+$ for the root lattice and its nonnegative semigroup generated by the positive roots $\boldsymbol{R}^+$

$$\mathcal{Q} = \mathrm{Span}_{\mathbb{Z}}(\boldsymbol{R}), \quad \mathcal{Q}^+ = \mathrm{Span}_{\mathbb{N}}(\boldsymbol{R}^+), \tag{2.1}$$

and we write $\mathcal{P}$ and $\mathcal{P}^+$ for the weight lattice its nonnegative cone of dominant weights

$$\mathcal{P} = \{\lambda \in \mathbf{E} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ \forall \alpha \in \boldsymbol{R}\}, \tag{2.2a}$$
$$\mathcal{P}^+ = \{\lambda \in \mathbf{E} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{N},\ \forall \alpha \in \boldsymbol{R}^+\}, \tag{2.2b}$$

where we have introduced the coroot $\alpha^\vee \equiv 2\alpha/\langle \alpha, \alpha \rangle$. The algebra of (trigonometric) polynomials on the Weyl alcove

$$\mathbf{A} = \{\xi \in \mathbf{E} \mid 0 < \langle \xi, \alpha \rangle < 2\pi,\ \forall \alpha \in \boldsymbol{R}^+\} \tag{2.3}$$

is spanned by the basis of the monomial symmetric functions

$$m_\lambda(\xi) = \frac{1}{|W_\lambda|} \sum_{w \in W} e^{i\langle \lambda, \xi_w \rangle}, \qquad \lambda \in \mathcal{P}^+, \tag{2.4}$$

where $W \subset \mathrm{GL}(\mathbf{E})$ denotes the Weyl group of the root system $\boldsymbol{R}$, $\xi_w \equiv w(\xi)$, and $|W_\lambda|$ stands for the order of the stabilizer subgroup $W_\lambda = \{w \in W \mid w(\lambda) = \lambda\}$.

2.2. **Factorized Weight Functions.** We will now introduce a class of smooth weight functions on the Weyl alcove $\mathbf{A}$ that factorize over the root system $\boldsymbol{R}$. To this end we write $\boldsymbol{R}_0 = \{\alpha \in \boldsymbol{R} \mid 2\alpha \not\in \boldsymbol{R}\}$ and $\boldsymbol{R}_1 = \{\alpha \in \boldsymbol{R} \mid \frac{\alpha}{2} \not\in \boldsymbol{R}\}$. (So for a reduced root system one has that $\boldsymbol{R}_0 = \boldsymbol{R}_1 = \boldsymbol{R}$ and for the nonreduced root system $\boldsymbol{R} = BC_N$ one has that $\boldsymbol{R}_0 = C_N$ and $\boldsymbol{R}_1 = B_N$.) The weight functions under consideration are of the form

$$\hat{\Delta}(\xi) = \frac{1}{\hat{\mathcal{C}}(\xi)\hat{\mathcal{C}}(-\xi)}, \tag{2.5a}$$

with

$$\hat{\mathcal{C}}(\xi) = \prod_{\alpha \in \boldsymbol{R}_1^+} \hat{c}_{|\alpha|}(e^{-i\langle \alpha, \xi \rangle}), \tag{2.5b}$$

where it assumed that the c-functions $\hat{c}_{|\alpha|}(z)$ building $\hat{\mathcal{C}}(\xi)$ (2.5b) depend only on the length of the root α (so $\hat{c}_{|\alpha|}(z) = \hat{c}_{|\beta|}(z)$ if α and β lie on the same Weyl-orbit). For technical reasons, we will furthermore assume that these c-functions $\hat{c}_{|\alpha|}(z)$ are *(i)* analytic and zero-free on a closed disc $\mathbb{D}_\varrho = \{z \in \mathbb{C} \mid |z| \leq \varrho\}$ of radius $\varrho > 1$, *(ii)* normalized such that $\hat{c}_{|\alpha|}(0) = 1$, and *(iii)* real-valued for $z \in \mathbb{R}$ (so $\hat{\mathcal{C}}(-\xi) = \overline{\hat{\mathcal{C}}(\xi)}$).

2.3. **Gram-Schmidt Orthogonalization.** The technical conditions on the c-functions ensure that $\hat{\Delta}(\xi)$ (2.5a), (2.5b) defines a smooth positive weight function on $\mathbf{A}$ (which extends analytically to a Weyl-group invariant function on $\mathbf{E}$). We employ this weight function to endow the space of trigonometric polynomials on the Weyl alcove with an inner product structure via embedding in the Hilbert space $L^2(\mathbf{A}, \hat{\Delta}|\delta|^2\, \mathrm{d}\xi)$:

$$(f, g)_{\hat{\Delta}} = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})} \int_{\mathbf{A}} f(\xi)\overline{g(\xi)}\, \hat{\Delta}(\xi)\, |\delta(\xi)|^2 \mathrm{d}\xi, \qquad \forall f, g \in L^2(\mathbf{A}, \hat{\Delta}|\delta|^2\, \mathrm{d}\xi), \tag{2.6}$$

where $\overline{g(\xi)}$ stands for the complex conjugate of $g(\xi)$, $\mathrm{Vol}(\mathbf{A}) = \int_{\mathbf{A}} \mathrm{d}\xi$, and $\delta(\xi)$ denotes the Weyl denominator

$$\delta(\xi) = \prod_{\alpha \in \boldsymbol{R}_0^+} (e^{i\langle \alpha, \xi \rangle/2} - e^{-i\langle \alpha, \xi \rangle/2}). \tag{2.7}$$

Let $\succeq$ be a (partial) order of the dominant weights $\mathcal{P}^+$ refining the *dominance partial order*

$$\lambda \geqslant \mu \iff \lambda - \mu \in \mathcal{Q}^+ \tag{2.8}$$

such that the highest-weight spaces $\mathrm{Span}\{m_\mu\}_{\mu\in\mathcal{P}^+,\mu\preceq\lambda}$ remain finite-dimensional for all $\lambda \in \mathcal{P}^+$. By applying the Gram-Schmidt process to the partially ordered monomial basis $\{m_\lambda\}_{\lambda\in\mathcal{P}^+}$, we construct a normalized basis $\{P_\lambda\}_{\lambda\in\mathcal{P}^+}$ of $L^2(\mathbf{A}, \hat{\Delta}|\delta|^2\,\mathrm{d}\xi)$ given by trigonometric polynomials of the form

$$P_\lambda(\xi) = \sum_{\mu\in\mathcal{P}^+,\,\mu\preceq\lambda} a_{\lambda\mu} m_\mu(\xi), \qquad \lambda \in \mathcal{P}^+, \tag{2.9a}$$

with coefficients $a_{\lambda\mu} \in \mathbb{C}$ such that

$$(P_\lambda, P_\mu)_{\hat{\Delta}} = \begin{cases} 0 & \text{if } \mu \prec \lambda, \\ 1 & \text{if } \mu = \lambda \end{cases} \tag{2.9b}$$

(where $a_{\lambda\lambda} > 0$ by convention). The Gram-Schmidt process guarantees that the polynomials P_λ, $\lambda \in \mathcal{P}^+$ are orthogonal when comparable in the (partial) order $\succeq$ (i.e. $(P_\lambda, P_\mu)_{\hat{\Delta}} = 0$ when $\lambda \succ \mu$ or $\lambda \prec \mu$). Hence, a sufficient condition to ensure that our polynomials form an orthonormal basis of the Hilbert space $L^2(\mathbf{A}, \hat{\Delta}|\delta|^2\,\mathrm{d}x)$ is to require the refinement $\succeq$ of the dominance order $\geqslant$ to be a *linear* ordering of $\mathcal{P}^+$. (The fact that the polynomials in Eqs. (2.9a), (2.9b) form a complete set in $L^2(\mathbf{A}, \hat{\Delta}|\delta|^2\,\mathrm{d}x)$ is a consequence of the Stone-Weierstrass theorem.) In general, the orthonormal basis in question depends on the choice of such linear refinement. It will turn out below, however, that for our principal applications the c-functions are such that the orthogonality is already guaranteed when taking for $\succeq$ simply the dominance ordering $\geqslant$ (2.8) itself (in other words, in such case the construction results to be independent of the choice of the linear refinement). From now on we will always assume that we have fixed a sufficiently fine (partial) ordering $\succeq$ so as to guarantee that the basis $\{P_\lambda\}_{\lambda\in\mathcal{P}^+}$ be *orthogonal* (i.e. $(P_\lambda, P_\mu)_{\hat{\Delta}} = 0$ when $\lambda \neq \mu$).

2.4. **Weyl Characters.** The simplest example of the above construction is the special case with unit c-functions, i.e., with $\hat{c}_{|\alpha|}(z) = 1$, $\forall \alpha \in \boldsymbol{R}_1^+$. The weight function then becomes of the form $\hat{\Delta}(\xi) = 1$ and the Gram-Schmidt process turns out to be independent of the choice of the refinement $\succeq$ of $\geqslant$ (i.e. in this case we may take $\succeq$ to be equal to $\geqslant$ without restriction). The corresponding orthonormal polynomials $P_\lambda(\xi)$ amount to the celebrated Weyl characters [M3, M4]

$$P_\lambda(\xi) = \chi_\lambda(\xi) \equiv \delta^{-1}(\xi) \sum_{w\in W} (-1)^w\, e^{i\langle \rho+\lambda, \xi_w\rangle}, \qquad \lambda \in \mathcal{P}^+, \tag{2.10}$$

where $(-1)^w \equiv \det(w)$ and $\rho \equiv \frac{1}{2}\sum_{\alpha\in\boldsymbol{R}_0^+} \alpha$.

For later use, it will actually be convenient to extend the definition of the Weyl characters $\chi_\lambda(\xi)$ in Eq. (2.10) to the case of nondominant weights λ. It is immediate from this definition that for $\lambda \in \mathcal{P} \setminus \mathcal{P}^+$

$$\chi_\lambda(\xi) = \begin{cases} (-1)^{w_{\rho+\lambda}} \chi_{w_{\rho+\lambda}(\rho+\lambda)-\rho}(\xi) & \text{if } |W_{\rho+\lambda}| = 1, \\ 0 & \text{if } |W_{\rho+\lambda}| > 1, \end{cases} \tag{2.11}$$

where, for $\mu \in \mathcal{P}$ regular, $w_\mu \in W$ denotes the unique Weyl group element such that $w_\mu(\mu) \in \mathcal{P}^+$.

3. Discrete (Pseudo) Laplacians on the Weyl Chamber

In this section we associate a commuting system of discrete pseudo Laplacians on $\mathcal{P}^+$ to our orthonormal polynomials $P_\lambda(\xi)$.

3.1. **Fourier Transform.** Let $\mathcal{H}$ be the Hilbert space $l^2(\mathcal{P}^+)$ of square-summable functions over the dominant cone $\mathcal{P}^+$ equipped with the standard inner product

$$(f,g)_{\mathcal{H}} = \sum_{\lambda\in\mathcal{P}^+} f_\lambda \overline{g_\lambda} \qquad (f,g\in l^2(\mathcal{P}^+)), \tag{3.1a}$$

and let $\hat{\mathcal{H}}$ be the Hilbert space $L^2(\mathbf{A},\mathrm{d}\xi)$ of square-integrable functions over the Weyl alcove equipped with the normalized inner product

$$(\hat{f},\hat{g})_{\hat{\mathcal{H}}} = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi \qquad (\hat{f},\hat{g}\in L^2(\mathbf{A},\mathrm{d}\xi)). \tag{3.1b}$$

By construction, the functions

$$\Psi_\lambda(\xi) = \hat{\Delta}^{1/2}(\xi)\delta(\xi)P_\lambda(\xi), \qquad \lambda\in\mathcal{P}^+ \tag{3.2}$$

form an orthonormal basis of $\hat{\mathcal{H}}$. As a result, the mapping $\mathcal{F}:\mathcal{H}\mapsto\hat{\mathcal{H}}$ given by $\phi_\lambda \xrightarrow{\mathcal{F}} \hat{\phi}(\xi)$ with

$$\begin{aligned}\hat{\phi}(\xi) &= (\phi,\Psi(\xi))_{\mathcal{H}} \\ &= \sum_{\lambda\in\mathcal{P}^+}\phi_\lambda\overline{\Psi_\lambda(\xi)}\end{aligned} \tag{3.3a}$$

constitutes a unitary Hilbert space isomorphism between $\mathcal{H}$ and $\hat{\mathcal{H}}$. The inverse mapping $\mathcal{F}^{-1}:\hat{\mathcal{H}}\mapsto\mathcal{H}$ takes the form $\hat{\phi}(\xi)\xrightarrow{\mathcal{F}^{-1}}\phi_\lambda$ with

$$\begin{aligned}\phi_\lambda &= (\hat{\phi},\overline{\Psi}_\lambda)_{\hat{\mathcal{H}}} \\ &= \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}}\hat{\phi}(\xi)\Psi_\lambda(\xi)\mathrm{d}\xi.\end{aligned} \tag{3.3b}$$

We will refer to $\mathcal{F}$ as the *Fourier transform* associated to the polynomials $P_\lambda(\xi)$. In the simplest case with unit c-functions, the wave functions amount to plane waves (cf. Section 2.4)

$$\Psi^{(0)}_\lambda(\xi) = \sum_{w\in W}(-1)^w\, e^{i\langle\rho+\lambda,\xi_w\rangle}. \tag{3.4}$$

The corresponding Fourier transform reduces to the conventional Fourier transform $\mathcal{F}^{(0)}:\mathcal{H}\mapsto\hat{\mathcal{H}}$ of the form $\phi_\lambda\xrightarrow{\mathcal{F}^{(0)}}\hat{\phi}(\xi)$ with

$$\hat{\phi}(\xi) = \sum_{w\in W}(-1)^w\sum_{\lambda\in\mathcal{P}^+}\phi_\lambda e^{-i\langle\rho+\lambda,\xi_w\rangle}. \tag{3.5a}$$

The inverse transform $(\mathcal{F}^{(0)})^{-1}:\hat{\mathcal{H}}\mapsto\mathcal{H}$ is then given by $\hat{\phi}(\xi)\xrightarrow{(\mathcal{F}^{(0)})^{-1}}\phi_\lambda$ with

$$\phi_\lambda = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\sum_{w\in W}(-1)^w\int_{\mathbf{A}}\hat{\phi}(\xi)e^{i\langle\rho+\lambda,\xi_w\rangle}\mathrm{d}\xi. \tag{3.5b}$$

3.2. **Pseudo Laplacians.** To the basis of fundamental weights $\omega_1, \ldots, \omega_N$ generating $\mathcal{P}^+$, we associate bounded multiplication operators $\hat{E}_1, \ldots, \hat{E}_N$ in $\hat{\mathcal{H}}$ of the form

$$\hat{E}_r(\xi) = \sum_{\nu \in W(\omega_r)} \exp(i\langle \nu, \xi \rangle), \qquad r = 1, \ldots, N, \tag{3.6}$$

where the sum is over all weights in the Weyl orbit of ω_r. The pullbacks of $\hat{E}_1, \ldots, \hat{E}_N$ with respect to the Fourier transform $\mathcal{F}$ define an integrable system of bounded commuting operators in $\mathcal{H}$

$$L_r = \mathcal{F}^{-1} \circ \hat{E}_r \circ \mathcal{F}, \qquad r = 1, \ldots, N. \tag{3.7}$$

We will refer to the commutative algebra $\mathbb{R}[L_1, \ldots, L_N]$ generated by these operators as the (algebra of) *discrete pseudo Laplacians* associated to the polynomials $P_\lambda(\xi)$. It is immediate from its construction as the pullback of a multiplication operator in $\hat{\mathcal{H}}$ (cf. Eq. (3.7)) that the pseudo Laplacian L_r has a purely absolutely continuous spectrum in $\mathcal{H}$ given by the compact set $\sigma(L_r) = \{\hat{E}_r(\xi) \mid \xi \in \overline{\mathbf{A}}\} \subset \mathbb{C}$. By acting with both sides of the operator equality $L_r \mathcal{F}^{-1} = \mathcal{F}^{-1} \hat{E}_r$ on (the complex conjugate of) an arbitrary element $\hat{\phi} \in \hat{\mathcal{H}}$, we get

$$L_r (\Psi_\lambda, \hat{\phi})_{\hat{\mathcal{H}}} = (\hat{E}_r \Psi_\lambda, \hat{\phi})_{\hat{\mathcal{H}}}, \qquad \forall \hat{\phi} \in \hat{\mathcal{H}}. \tag{3.8a}$$

In other words, the functions $\Psi_\lambda(\xi)$ form a complete (as $\mathcal{F} : \mathcal{H} \to \hat{\mathcal{H}}$ is a Hilbert space isomorphism) set of generalized joint eigenfunctions of our pseudo Laplacians, i.e. formally

$$L_r \Psi_\lambda(\xi) = \hat{E}_r(\xi) \Psi_\lambda(\xi). \tag{3.8b}$$

Here $\xi \in \overline{\mathbf{A}}$ plays the role of the spectral parameter and the weight $\lambda \in \mathcal{P}^+$ is interpreted as the discrete geometric variable (i.e. the position variable).

Note. i. Below we will sometimes write formal equalities of the form in Eq. (3.8b) that admit a rigorous interpretation of the form in Eq. (3.8a) upon taking the inner product (smearing) with an arbitrary (stationary) wave packet $\hat{\phi} \in \hat{\mathcal{H}}$.

Note. ii. In general the Laplacian L_r is not self-adjoint. Indeed, the adjoint L_r^* is given by L_s with $\omega_s = -w_0(\omega_r)$, where w_0 denotes the longest element of the Weyl group W (i.e., the unique Weyl group element w_0 such that $w_0(\mathbf{A}) = -\mathbf{A}$). Thus L_r is self-adjoint if and only if $w_0(\omega_r) = -\omega_r$.

3.3. **Localization.** Let $\phi : \mathcal{P}^+ \to \mathbb{C}$ be square-summable a lattice function. The action of L_r on ϕ is of the form

$$L_r \phi_\lambda = \sum_{\mu \in \mathcal{P}^+} a_{\lambda\mu;r} \phi_\mu, \tag{3.9}$$

for certain coefficients $a_{\lambda\mu;r} \in \mathbb{C}$. We will now show that in fact only a finite number of these coefficients is nonzero.

Proposition 3.1 (Localization)**.** *The action of the pseudo Laplacian L_r on $\phi \in \mathcal{H}$ is of the form*

$$L_r \phi_\lambda = \sum_{\mu \in \mathcal{P}^+_{\lambda;r}} a_{\lambda\mu;r} \phi_\mu, \qquad a_{\lambda\mu;r} \in \mathbb{C}, \tag{3.10a}$$

where

$$\mathcal{P}^+_{\lambda;r} = \{\mu \in \mathcal{P}^+ \mid \mu \preceq \lambda + \omega_r \text{ and } \mu - w_0(\omega_r) \succeq \lambda\}. \tag{3.10b}$$

Proof. From the triangularity of the monomial expansion of $P_\lambda(\xi)$ it is immediate that

$$m_{\omega_r}(\xi)P_\lambda(\xi) = \sum_{\substack{\mu\in\mathcal{P}^+\\ \mu\preceq\lambda+\omega_r}} a_{\lambda\mu;r}P_\mu(\xi), \qquad a_{\lambda\mu;r}\in\mathbb{C}.$$

The orthonormality furthermore implies that

$$a_{\lambda\mu;r} = (m_{\omega_r}P_\lambda, P_\mu)_{\hat{\Delta}} = (P_\lambda, m_{\omega_s}P_\mu)_{\hat{\Delta}} = \overline{a_{\mu\lambda;s}},$$

with $\omega_s = -w_0(\omega_r)$ (cf. Note *ii.* above). Hence

$$a_{\lambda\mu;r}\neq 0 \Rightarrow \mu\in\mathcal{P}^+_{\lambda;r}.$$

Since $\hat{E}_r(\xi) = m_{\omega_r}(\xi)$ and $\Psi_\lambda(\xi) = \hat{\Delta}^{1/2}(\xi)\delta(\xi)P_\lambda(\xi)$, we conclude that

$$\hat{E}_r(\xi)\Psi_\lambda(\xi) = \sum_{\mu\in\mathcal{P}^+_{\lambda;r}} a_{\lambda\mu;r}\Psi_\mu(\xi).$$

Taking the innerproduct with an arbitrary wave packet $\hat{\phi}\in\hat{\mathcal{H}}$ and comparison with the eigenvalue equation in Eq. (3.8a) entails that

$$L_r(\Psi_\lambda,\hat{\phi})_{\hat{\mathcal{H}}} = \sum_{\mu\in\mathcal{P}^+_{\lambda;r}} a_{\lambda\mu;r}(\Psi_\mu,\hat{\phi})_{\hat{\mathcal{H}}}, \qquad \forall\hat{\phi}\in\hat{\mathcal{H}},$$

i.e. formally (cf. Note *i.* above)

$$L_r\Psi_\lambda(\xi) = \sum_{\mu\in\mathcal{P}^+_{\lambda;r}} a_{\lambda\mu;r}\Psi_\mu(\xi).$$

The proposition then follows by the completeness of the generalized eigenfunctions $\Psi_\lambda(\xi)$, $\xi\in\overline{\mathbf{A}}$ in the Hilbert space $\mathcal{H}$ (i.e. by the fact that the Fourier transform $\mathcal{F}$ (3.3a), (3.3b) constitutes a unitary Hilbert space isomorphism between $\mathcal{H}$ and $\hat{\mathcal{H}}$). □

A priori the cardinality of the set $\mathcal{P}^+_{\lambda;r}$ may be unbounded as a function of $\lambda\in\mathcal{P}^+$. Hence, in general our pseudo Laplacians need *not* be difference operators. If the ordering of the dominant weights $\succeq$ coincides with the dominance order $\geqslant$, however, then it follows from the definition in Eq. (2.8) that the size of the set $\mathcal{P}^+_{\lambda;r}$ is bounded by the number of weights in the interval $\{\nu\in\mathcal{P}\mid w_0(\omega_r)\leqslant\nu\leqslant\omega_r\}$. Consequently, in this situation our pseudo Laplacian L_r is actually a difference operator in $\mathcal{H}$. (We will refer in such case to L_r as a *discrete Laplacian* as opposed to merely a pseudo Laplacian.)

Proposition 3.2 (Discrete Laplacians)**.** *When our ordering $\succeq$ coincides with the dominance ordering $\geqslant$ (2.8), then the pseudo Laplacians in $\mathbb{R}[L_1,\ldots,L_N]$ are discrete difference operators in $\mathcal{H}$.*

In the case of unit c-functions (cf. Section 2.4), our discrete Laplacians $L_1,\ldots,L_N$ amount to conventional free Laplacians $L_1^{(0)},\ldots,L_N^{(0)}$ over the dominant cone $\mathcal{P}^+$.

Proposition 3.3 (Free Laplacians)**.** *If $\hat{c}_{|\alpha|}(z)=1$, $\forall\alpha\in\boldsymbol{R}_1^+$, then our discrete Laplacians L_r reduce to the free Laplacians*

$$L_r^{(0)}\phi_\lambda = \sum_{\nu\in W(\omega_r)}\phi_{\lambda+\nu}, \qquad r=1,\ldots,N,$$

with the boundary condition that for $\mu \in \mathcal{P} \setminus \mathcal{P}^+$

$$\phi_\mu = \begin{cases} (-1)^{w_{\rho+\mu}} \phi_{w_{\rho+\mu}(\rho+\mu)-\rho} & \text{if } |W_{\rho+\mu}| = 1, \\ 0 & \text{if } |W_{\rho+\mu}| > 1 \end{cases}$$

(where $w_{\rho+\mu}$ denotes the Weyl permutation taking the regular weight $\rho+\mu$ to the dominant cone).

Proof. As pointed out in Section 2.4, the case of unit c-functions corresponds to orthonormal polynomials $P_\lambda(\xi)$ given by the Weyl characters $\chi_\lambda(\xi)$. It is immediate from the explicit expression for χ_λ in Eq. (2.10) that the Weyl characters satisfy the well-known recurrence relations

$$m_{\omega_r}(\xi)\chi_\lambda(\xi) = \sum_{\nu \in W(\omega_r)} \chi_{\lambda+\nu}(\xi).$$

Starting from these recurrence relations, the proposition readily follows by repeating the arguments in the proof of Proposition 3.1. The boundary condition stems from the property (2.11) of the Weyl characters. □

By Proposition 3.1, the eigenvalue equations in Eq. (3.8b) take the form

$$\sum_{\mu \in \mathcal{P}^+_{\lambda;r}} a_{\lambda\mu;r} \Psi_\mu(\xi) = \hat{E}_r(\xi)\Psi_\lambda(\xi), \qquad 1, \ldots, N, \tag{3.11a}$$

or equivalently

$$\sum_{\mu \in \mathcal{P}^+_{\lambda;r}} a_{\lambda\mu;r} P_\mu(\xi) = \hat{E}_r(\xi)P_\lambda(\xi), \qquad 1, \ldots, N \tag{3.11b}$$

(upon dividing out the trivial overall normalization factor $\delta(\xi)/\sqrt{\hat{\mathcal{C}}(\xi)\hat{\mathcal{C}}(-\xi)}$ on both sides). The latter equations admit an alternative interpretation as a system of recurrence relations (or Pieri formulas) for the polynomials $P_\lambda(\xi)$.

Note. The above construction of the discrete (pseudo) Laplacians has its origin in the works of Macdonald [M1, M3, M4]. Specifically, for $c_{|\alpha|}(z) = (1 - t_{|\alpha|}z)$ with $-1 < t_{|\alpha|} < 1$ the polynomials $P_\lambda(\xi)$ (2.9a), (2.9b) amount to (the parameter deformations of) Macdonald's zonal spherical functions on p-adic Lie groups [M1, M3]. The algebra of discrete Laplacians $\mathbb{R}[L_1, \ldots, L_N]$ corresponds in this case to the K-spherical Hecke algebra of the p-adic Lie group. When $c_{|\alpha|}(z)$ is given by a q-shifted factorial (cf. Eq. (6.1) below), then the polynomials $P_\lambda(\xi)$ specialize to the Macdonald polynomials [M2, M3, M4]. The discrete Laplacians appear in this context in Cherednik's double affine Hecke algebra as "coordinate multiplication operators" dual to Macdonald's difference operators [C2, M4].

4. Time-Dependent Scattering Theory

In this section we determine the wave operators and scattering operator associated to our discrete pseudo Laplacians. For background literature on scattering theory the reader is referred to e.g. Refs. [RS, P, T].

4.1. **Plane Wave Asymptotics.** The dominant Weyl chamber is given by the open convex cone

$$\mathbf{C}^+ = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \alpha \rangle > 0,\ \forall \alpha \in \boldsymbol{R}^+\}. \tag{4.1}$$

We will now describe the asymptotics of the wave function $\Psi_\lambda(\xi)$ (3.2) diagonalizing the pseudo Laplacians $L_1, \ldots, L_N$ (3.7) for λ deep in the Weyl chamber, i.e., for λ growing to infinity in such a way that $\langle \lambda, \alpha^\vee \rangle \to +\infty$ for all positive roots $\alpha \in \boldsymbol{R}^+$.

To this end we define for $\lambda \in \mathcal{P}^+$

$$m(\lambda) \equiv \min_{\alpha \in \boldsymbol{R}^+} \langle \lambda, \alpha^\vee \rangle. \tag{4.2}$$

In previous work, it was shown that the strong L^2-asymptotics of the polynomials $P_\lambda(\xi)$ for $m(\lambda) \to \infty$ is given by [R4, D3, D4]

$$P_\lambda^\infty(\xi) = \delta^{-1}(\xi) \sum_{w \in W} (-1)^w \hat{\mathcal{C}}(\xi_w) e^{i\langle \rho+\lambda, \xi_w \rangle}. \tag{4.3}$$

More precisely, one has that

$$\|P_\lambda - P_\lambda^\infty\|_{\hat{\Delta}} = O(e^{-\epsilon\, m(\lambda)}) \quad \text{as} \;\; m(\lambda) \longrightarrow \infty, \tag{4.4}$$

where $\|\cdot\|_{\hat{\Delta}} \equiv (\cdot,\cdot)_{\hat{\Delta}}^{1/2}$ and $\epsilon > 0$ denotes a decay rate that depends on the radius $\varrho > 1$ of the analyticity disc $\mathbb{D}_\varrho$ of the c-functions $\hat{c}_{|\alpha|}(z)$ (see the technical assumptions in Section 2).

The idea of the proof in [R4, D3, D4] of this exponential convergence goes along the following lines. Firstly, a direct (constant term) computation reveals that

$$\langle P_\lambda^\infty, m_\mu \rangle_{\hat{\Delta}} = \begin{cases} 0 & \text{if } \mu \prec \lambda, \\ 1 & \text{if } \mu = \lambda. \end{cases} \tag{4.5a}$$

Next, we denote by $P_\lambda^{(m(\lambda))}(\xi)$ the polynomial approximation of the asymptotic function $P_\lambda^\infty(\xi)$ obtained by replacing the overall c-function $\hat{\mathcal{C}}(\xi)$ by its Taylor polynomial of degree $m(\lambda)$. Then a combinatorial analysis shows that this polynomial approximation expands triangularly on the basis of monomial symmetric functions

$$P_\lambda^{(m(\lambda))}(\xi) = m_\lambda(\xi) + \sum_{\mu \in \mathcal{P}^+, \mu \prec \lambda} b_{\lambda\mu} m_\mu(\xi) \tag{4.5b}$$

(for certain coefficients $b_{\lambda\mu} \in \mathbb{C}$). Moreover, the analyticity requirements on the c-functions $\hat{c}_{|\alpha|}(z)$ guarantee that

$$P_\lambda^\infty(\xi) = P_\lambda^{m(\lambda)}(\xi) + O(e^{-\epsilon m(\lambda)}) \tag{4.5c}$$

(this is because the technical conditions ensure that the Taylor coefficients of the c-function $\hat{c}_{|\alpha|}(z)$ decay exponentially fast). From Eqs. (4.5a)-(4.5c) one concludes that—up to an $O(e^{-\epsilon m(\lambda)})$ error term—the asymptotic function $P_\lambda^\infty(\xi)$ amounts to a monic polynomial obtained by performing the Gram-Schmidt process on the monomial symmetric basis with respect to the inner product $(\cdot,\cdot)_{\hat{\Delta}}$. In other words, the asymptotic functions coincide up to exponentially decaying error terms with the monic versions of the polynomials $P_\lambda(\xi)$ defined in Eqs. (2.9a), (2.9b). The convergence in Eq. (4.4) now follows from the fact that the orthonormalized polynomials $P_\lambda(\xi)$ are asymptotically monic: $a_{\lambda\lambda} = 1 + O(e^{-\epsilon m(\lambda)})$. (This estimate for the leading coefficient in the monomial expansion of $P_\lambda(\xi)$ follows starting from the equality $a_{\lambda\lambda} = \langle P_\lambda, P_\lambda^\infty \rangle_{\hat{\Delta}}$, upon substituting (4.5c) and expanding the polynomial

part $P_\lambda^{m(\lambda)}(\xi)$ in terms of the orthonormalized polynomials $P_\mu(\xi)$, $\mu \preceq \lambda$, taking into account the orthogonality (2.9b).)

The asymptotic estimate in Eq. (4.4) for the polynomials $P_\lambda(\xi)$ immediately gives rise to the following plane wave asymptotics for the wave functions $\Psi_\lambda(\xi)$ (3.2):

$$\Psi_\lambda^\infty(\xi) = \hat{\Delta}^{1/2}(\xi)\delta(\xi)P_\lambda^\infty(\xi) \tag{4.6a}$$

$$= \sum_{w\in W} (-1)^w \hat{S}_w^{1/2}(\xi) e^{i\langle \rho+\lambda, \xi_w\rangle}, \tag{4.6b}$$

where

$$\hat{S}_w(\xi) = \frac{\hat{\mathcal{C}}(\xi_w)}{\hat{\mathcal{C}}(-\xi_w)} \tag{4.6c}$$

$$= \prod_{\alpha\in \boldsymbol{R}_1^+\cap w^{-1}(\boldsymbol{R}_1^+)} \hat{s}_{|\alpha|}(\langle\alpha,\xi\rangle) \prod_{\alpha\in \boldsymbol{R}_1^+\cap w^{-1}(-\boldsymbol{R}_1^+)} \overline{\hat{s}_{|\alpha|}(\langle\alpha,\xi\rangle)}, \tag{4.6d}$$

with

$$\hat{s}_{|\alpha|}(\langle\alpha,\xi\rangle) = \frac{\hat{c}_{|\alpha|}(e^{-i\langle\alpha,\xi\rangle})}{\hat{c}_{|\alpha|}(e^{i\langle\alpha,\xi\rangle})} \tag{4.6e}$$

(so $\overline{\hat{s}_{|\alpha|}(\langle\alpha,\xi\rangle)} = \hat{s}_{|\alpha|}(-\langle\alpha,\xi\rangle) = \hat{s}_{|\alpha|}^{-1}(\langle\alpha,\xi\rangle)$ and $|\hat{s}_{|\alpha|}(\langle\alpha,\xi\rangle)| = 1$).

Theorem 4.1 (Plane Wave Asymptotics)**.** *The wave function Ψ_λ tends to the plane waves Ψ_λ^∞ for λ deep in the Weyl chamber:*

$$\|\Psi_\lambda - \Psi_\lambda^\infty\|_{\hat{\mathcal{H}}} = O(e^{-\epsilon\, m(\lambda)}) \quad \textit{as} \;\; m(\lambda) \longrightarrow \infty,$$

where $\|\cdot\|_{\hat{\mathcal{H}}} \equiv (\cdot,\cdot)_{\hat{\mathcal{H}}}^{1/2}$.

We see from Theorem 4.1 that the asymptotics of the wave functions $\Psi_\lambda(\xi)$ for λ deep in the Weyl chamber is given by an anti-symmetric combination of plane waves $e^{i\langle\lambda,\xi\rangle}$ with phase-factors that factorize over the root system in terms of one-dimensional c-functions.

4.2. **Scattering and Wave Operators.** For any *real* multiplication operator $\hat{E}(\xi) \subset \mathbb{R}[\hat{E}_1(\xi),\ldots,\hat{E}_N(\xi)]$, let $L = \mathcal{F}^{-1}\circ\hat{E}\circ\mathcal{F}$ and let $L^{(0)} = (\mathcal{F}^{(0)})^{-1}\circ\hat{E}\circ\mathcal{F}^{(0)}$. In other words, the operators $L \subset \mathbb{R}[L_1,\ldots,L_N]$ and $L^{(0)} \subset \mathbb{R}[L_1^{(0)},\ldots,L_N^{(0)}]$ are *self-adjoint* (pseudo) Laplacians in $\mathcal{H}$ such that (formally)

$$L\Psi_\lambda(\xi) = \hat{E}(\xi)\Psi_\lambda(\xi) \quad \text{and} \quad L^{(0)}\Psi_\lambda^{(0)}(\xi) = \hat{E}(\xi)\Psi_\lambda^{(0)}(\xi). \tag{4.7}$$

(So the spectrum of L and $L^{(0)}$ in $\mathcal{H}$ is absolutely continuous and given by the compact interval $\sigma(L) = \sigma(L^{(0)}) = \{\hat{E}(\xi) \mid \xi \in \overline{\mathbf{A}}\}$.) We will now describe the scattering of the interacting dynamics generated by the discrete pseudo Laplacian L with respect to the free dynamics generated by the discrete Laplacian $L^{(0)}$. Let us to this end define the regular sector of the Weyl alcove as

$$\mathbf{A}_{\mathrm{reg}} = \{\xi \in \mathbf{A} \mid \langle \nabla\hat{E}, \alpha\rangle \neq 0,\ \forall \alpha \in \boldsymbol{R}^+\}. \tag{4.8}$$

Due to the analyticity of $\hat{E}(\xi)$, the regular sector $\mathbf{A}_{\mathrm{reg}}$ is an open dense subset of the Weyl alcove $\mathbf{A}$. For every $\xi \in \mathbf{A}_{\mathrm{reg}}$, there exists now a unique Weyl group element $\hat{w}_\xi \in W$ such that $\hat{w}_\xi(\nabla\hat{E})$ lies in the dominant Weyl chamber $\mathbf{C}^+$ (4.1). Clearly, the Weyl-group valued function $\xi \to \hat{w}_\xi$ is constant on the connected components of $\mathbf{A}_{\mathrm{reg}}$ (by continuity). We are now in the position to define the

unitary multiplication operator $\hat{\mathcal{S}}_L : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$ (the so-called *scattering matrix*) via its restriction to the dense subspace of (say) smooth complex test functions with compact support in $\mathbf{A}_{\text{reg}}$:

$$(\hat{\mathcal{S}}_L \hat{\phi})(\xi) = \hat{S}_{\hat{w}_\xi}(\xi)\hat{\phi}(\xi) \qquad (\hat{\phi} \in C_0^\infty(\mathbf{A}_{\text{reg}})), \tag{4.9}$$

where $\hat{S}_w(\xi)$ is given by Eq. (4.6d).

The main result of this paper is the following explicit formula for the wave operators and the scattering operator in terms of the scattering matrix $\hat{\mathcal{S}}_L$ (4.9) and the Fourier transforms $\mathcal{F}$ (3.3a), (3.3b) and $\mathcal{F}^{(0)}$ (3.5a), (3.5b), thus relating the long-time asymptotics of *interacting dynamics* e^{itL} to that of the *free dynamics* $e^{itL^{(0)}}$. The proof, which is relegated to Section 5 below, consists of a stationary phase analysis based on the asymptotic formula for the wave functions in Theorem 4.1.

Theorem 4.2 (Wave Operators)**.** *The operator limits*

$$\Omega_\pm = s - \lim_{t\to\pm\infty} e^{itL} e^{-itL^{(0)}}$$

converge in the strong $\|\cdot\|_{\mathcal{H}}$*-norm topology (where* $\|\cdot\|_{\mathcal{H}} = (\cdot,\cdot)_{\mathcal{H}}^{1/2}$*), and the corresponding wave operators* $\Omega_\pm : \mathcal{H} \mapsto \mathcal{H}$ *are given by the unitary operators*

$$\begin{aligned} \Omega_+ &= \mathcal{F}^{-1} \circ \hat{\mathcal{S}}_L^{-1/2} \circ \mathcal{F}^{(0)}, \\ \Omega_- &= \mathcal{F}^{-1} \circ \hat{\mathcal{S}}_L^{1/2} \circ \mathcal{F}^{(0)}. \end{aligned}$$

Corollary 4.3 (Scattering Operator)**.** *The scattering operator* $\mathcal{S}_L : \mathcal{H} \mapsto \mathcal{H}$ *for the self-adjoint discrete pseudo Laplacian* $L \in \mathbb{R}[L_1, \ldots, L_N]$ *is given by the unitary operator*

$$\mathcal{S}_L \equiv \Omega_+^{-1}\Omega_- = (\mathcal{F}^{(0)})^{-1} \circ \hat{\mathcal{S}}_L \circ \mathcal{F}^{(0)}.$$

We see from Corollary 4.3 and Eqs. (4.6d), (4.9) that the scattering matrix for the self-adjoint discrete pseudo Laplacian L factorizes over the root system $\boldsymbol{R}$. For the type A root systems, Theorem 4.2 and Corollary 4.3 reproduce the results of Ruijsenaars in Ref. [R4].

5. Stationary Phase Analysis

In this section the fundamental formulas for the wave operators stated in Theorem 4.2 are proven. To this end we employ a stationary phase method that generalizes Ruijsenaars' approach in Ref. [R4] from the type A root systems to the case of arbitrary crystallographic root systems. Throughout this section the notational conventions of Sections 3 and 4 are adopted.

5.1. **Asymptotics of Wave Packets.** Let us introduce the *free wave packet* $\phi^{(0)}(t)$ and the *interacting wave packets* $\phi_\pm(t)$ of the form

$$\phi^{(0)}(t) = (\mathcal{F}^{(0)})^{-1} e^{-it\hat{E}} \hat{\phi}, \tag{5.1a}$$

$$\phi_\pm(t) = \mathcal{F}^{-1} e^{-it\hat{E}} \hat{\mathcal{S}}_L^{\mp 1/2} \hat{\phi}, \tag{5.1b}$$

or more explicitly

$$\phi^{(0)}_\lambda(t) = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\sum_{w\in W}(-1)^w\int_{\mathbf{A}} e^{i\langle\rho+\lambda,\xi_w\rangle-it\hat{E}(\xi)}\hat{\phi}(\xi)\mathrm{d}\xi, \tag{5.2a}$$

$$\phi_{\pm,\lambda}(t) = \frac{1}{|W|\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}}\Psi_\lambda(\xi)e^{-it\hat{E}(\xi)}\hat{\mathcal{S}}_L^{\mp1/2}(\xi)\hat{\phi}(\xi)\mathrm{d}\xi, \tag{5.2b}$$

with $\hat{\phi}\in C_0^\infty(\mathbf{A}_{\mathrm{reg}})$. The following lemma states that the long-time asymptotics of the interacting wave packets $\phi_+(t)$ and $\phi_-(t)$ for $t\to+\infty$ and $t\to-\infty$, respectively, coincides with the corresponding asymptotics of the free wave packet $\phi^{(0)}(t)$.

Proposition 5.1 (Asymptotic Freedom). *For $t\to\pm\infty$, the difference between the interacting wave packet $\phi_\pm(t)$ and the free wave packet $\phi^{(0)}(t)$ tends to zero:*

$$\forall\kappa>0: \qquad \|\phi_\pm(t)-\phi^{(0)}(t)\|_{\mathcal{H}} = O(1/|t|^\kappa) \quad \text{as}\, t\to\pm\infty.$$

Before proving Proposition 5.1 (cf. below), let us first infer that Theorem 4.2 arises as an immediate consequence. Indeed, since the space of test functions $C_0^\infty(\mathbf{A}_{\mathrm{reg}})$ is dense in $\hat{\mathcal{H}}$ and the operators in question are unitary, to validate Theorem 4.2 it is sufficient to demonstrate that for $\phi=(\mathcal{F}^{(0)})^{-1}\hat{\phi}$ with $\hat{\phi}\in C_0^\infty(\mathbf{A}_{\mathrm{reg}})$

$$\lim_{t\to\pm\infty}\|e^{itL}e^{-itL^{(0)}}\phi-\Omega_\pm\phi\|_{\mathcal{H}}=0,$$

where $\Omega_\pm\equiv\mathcal{F}^{-1}\circ\hat{\mathcal{S}}_L^{\mp1/2}\circ\mathcal{F}^{(0)}$. From the unitarity of e^{itL} and the intertwining relations

$$e^{-itL^{(0)}}\circ(\mathcal{F}^{(0)})^{-1}=(\mathcal{F}^{(0)})^{-1}\circ e^{-it\hat{E}} \quad \text{and} \quad e^{-itL}\circ\mathcal{F}^{-1}=\mathcal{F}^{-1}\circ e^{-it\hat{E}},$$

it is clear that

$$\begin{aligned}\|e^{itL}e^{-itL^{(0)}}\phi-\Omega_\pm\phi\|_{\mathcal{H}} &= \|e^{-itL^{(0)}}(\mathcal{F}^{(0)})^{-1}\hat{\phi}-e^{-itL}\Omega_\pm(\mathcal{F}^{(0)})^{-1}\hat{\phi}\|_{\mathcal{H}}\\ &= \|\phi^{(0)}(t)-\phi_\pm(t)\|_{\mathcal{H}},\end{aligned}$$

whence the theorem follows from Proposition 5.1.

5.2. **The Classical Wave Packet.** To prove Proposition 5.1, we may assume without restricting generality that the test function $\hat{\phi}$ has in fact compact support inside a connected component of $\mathbf{A}_{\mathrm{reg}}$. In this situation there thus exists a unique Weyl group element $\hat{w}\in W$ such that $\hat{w}(\nabla\hat{E})$ lies inside the dominant Weyl chamber $\mathbf{C}^+$ (4.1) for all ξ in the support of $\hat{\phi}$. Let $\mathbf{V}_{\mathrm{clas}}\subset\mathbf{E}$ be an open bounded neighborhood of the compact range of classical wave-packet velocities $\mathrm{Ran}_{\hat{\phi}}(\nabla\hat{E})\equiv\{\nabla\hat{E}(\xi)\mid\xi\in\mathrm{Supp}(\hat{\phi})\}$ staying away from the walls of the Weyl chamber $\hat{w}^{-1}(\mathbf{C}^+)$ in the sense that there exists a lower-bound $\varepsilon>0$ such that $\langle\zeta_{\hat{w}},\alpha^\vee\rangle>\varepsilon$ for all $\zeta\in\mathbf{V}_{\mathrm{clas}}$ and all $\alpha\in\boldsymbol{R}^+$. We will now introduce a *classical wave packet* that is finitely supported on the following t-dependent region of the cone of dominant weights $\mathcal{P}^+$

$$\mathcal{P}^+_{\mathrm{clas}}(t)=\begin{cases}\{\lambda\in\mathcal{P}^+\mid\rho+\lambda\in t\hat{w}(\mathbf{V}_{\mathrm{clas}})\} & \text{for } t>0,\\ \{\lambda\in\mathcal{P}^+\mid\rho+\lambda\in tw_0\hat{w}(\mathbf{V}_{\mathrm{clas}})\} & \text{for } t<0.\end{cases} \tag{5.3}$$

Because of dimensional considerations, it is clear that the cardinality of the support $\mathcal{P}^+_{\mathrm{clas}}(t)$ grows at most polynomially in t

$$|\mathcal{P}^+_{\mathrm{clas}}(t)|=O(t^N) \qquad \text{for } |t|\to\infty. \tag{5.4}$$

The classical wave packet is defined as

$$\phi_\lambda^{(\mathrm{clas})}(t) = \tag{5.5}$$
$$\begin{cases} \frac{(-1)^{\hat{w}}}{|W|\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}} e^{i\langle\rho+\lambda,\xi_{\hat{w}}\rangle - it\hat{E}(\xi)}\hat{\phi}(\xi)\mathrm{d}\xi & \text{for } \lambda\in\mathcal{P}^+_{\mathrm{clas}}(t) \text{ and } t>0,\\ \frac{(-1)^{w_0\hat{w}}}{|W|\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}} e^{i\langle\rho+\lambda,\xi_{w_0\hat{w}}\rangle - it\hat{E}(\xi)}\hat{\phi}(\xi)\mathrm{d}\xi & \text{for } \lambda\in\mathcal{P}^+_{\mathrm{clas}}(t) \text{ and } t<0,\\ 0 & \text{otherwise.}\end{cases}$$

The next lemma compares the long-time asymptotics of the free wave packet $\phi^{(0)}(t)$ with that of the classical wave packet $\phi^{(\mathrm{clas})}(t)$.

Lemma 5.2. *For $t\to\pm\infty$, the difference between the free wave packet $\phi^{(0)}(t)$ and the classical wave packet $\phi^{(clas)}(t)$ tends to zero:*

$$\forall\kappa>0: \qquad \|\phi^{(0)}(t)-\phi^{(clas)}(t)\|_{\mathcal{H}} = O(1/|t|^\kappa) \quad \text{as } t\to\pm\infty.$$

Proof. It is immediate from the definitions of the wave packets under consideration that

$$\phi_\lambda^{(0)}(t)-\phi_\lambda^{(\mathrm{clas})}(t) = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\sum_{w\in\hat{W}}(-1)^w\int_{\mathbf{A}} e^{i\langle\rho+\lambda,\xi_w\rangle-it\hat{E}(\xi)}\hat{\phi}(\xi)\mathrm{d}\xi, \tag{5.6a}$$

with

$$\hat{W} \equiv \begin{cases} W\setminus\{\hat{w}\} & \text{if } \lambda\in\mathcal{P}^+_{\mathrm{clas}}(t) \text{ and } t>0,\\ W\setminus\{w_0\hat{w}\} & \text{if } \lambda\in\mathcal{P}^+_{\mathrm{clas}}(t) \text{ and } t<0,\\ W & \text{otherwise.}\end{cases} \tag{5.6b}$$

The proof of the lemma now hinges on a stationary phase estimate extracted from the Corollary of Theorem XI.14 in Ref. [RS, p. 38-39], which states that for any $k>0$ there exists a (positive) constant c_k such that

$$\left|\int_{\mathbf{A}} e^{i\langle\mathbf{x},\xi\rangle-it\hat{E}(\xi)}\hat{\phi}(\xi)\mathrm{d}\xi\right| \leq \frac{c_k}{(1+|\mathbf{x}|+|t|)^k} \tag{5.7a}$$

for all $\mathbf{x}\in\mathbf{E}$ and $t\in\mathbb{R}$ such that

$$\mathbf{x}\notin t\mathbf{V}_{\mathrm{clas}}. \tag{5.7b}$$

Indeed, invoking of the stationary phase estimate in Eqs. (5.7a), (5.7b) with $k>N/2$ and $\mathbf{x}=w^{-1}(\rho+\lambda)$, reveals that the norm of the difference between the wave packets given by Eqs. (5.6a), (5.6b) in the Hilbert space $\mathcal{H}$ is $O(1/|t|^{k-N/2})$ as $t\to\pm\infty$. (Notice in this connection that $w^{-1}(\rho+\lambda)\in t\mathbf{V}_{\mathrm{clas}}$ if and only if $\lambda\in\mathcal{P}^+_{\mathrm{clas}}(t)$ and $w\in W\setminus\hat{W}$.) $\square$

5.3. **The Asymptotic Wave Packet.** Let us define *asymptotic wave packets* $\phi_\pm^{(\infty)}(t)$ that are obtained from the interacting wave packets $\phi_\pm(t)$ upon replacing the Fourier kernel $\Psi_\lambda(\xi)$ by its plane wave asymptotics $\Psi_\lambda^{(\infty)}(\xi)$

$$\phi_{\pm,\lambda}^{(\infty)}(t) = \frac{1}{|W|\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}}\Psi_\lambda^{(\infty)}(\xi)e^{-it\hat{E}(\xi)}\hat{\mathcal{S}}_L^{\mp1/2}(\xi)\hat{\phi}(\xi)\mathrm{d}\xi, \tag{5.8a}$$

or more explicitly

$$\phi^{(\infty)}_{+,\lambda}(t) = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\sum_{w\in W}(-1)^w\int_{\mathbf{A}} e^{i\langle\rho+\lambda,\xi_w\rangle-it\hat{E}(\xi)}\frac{\hat{\mathcal{C}}(\xi_w)}{\hat{\mathcal{C}}(\xi_{\hat{w}})}\hat{\phi}(\xi)\mathrm{d}\xi, \quad (5.8b)$$

$$\phi^{(\infty)}_{-,\lambda}(t) = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\sum_{w\in W}(-1)^w\int_{\mathbf{A}} e^{i\langle\rho+\lambda,\xi_w\rangle-it\hat{E}(\xi)}\frac{\hat{\mathcal{C}}(\xi_w)}{\hat{\mathcal{C}}(\xi_{w_0\hat{w}})}\hat{\phi}(\xi)\mathrm{d}\xi. \quad (5.8c)$$

The next lemma states that the long-time behavior of the asymptotic wave packets is governed by the classical wave packet $\phi^{(\mathrm{clas})}(t)$ (5.5).

Lemma 5.3. *For $t\to\pm\infty$, the difference between the asymptotic wave packet $\phi^{(\infty)}_{\pm}(t)$ and the classical wave packet $\phi^{(clas)}(t)$ tends to zero:*

$$\forall\kappa>0: \qquad \|\phi^{(\infty)}_{\pm}(t)-\phi^{(clas)}(t)\|_{\mathcal{H}} = O(1/|t|^{\kappa}) \quad \text{as } t\to\pm\infty.$$

Proof. The proof of Lemma 5.2 applies verbatim, upon replacing $\phi^{(0)}(t)$ by $\phi^{(\infty)}_{\pm}(t)$ and the introduction of minor modifications in the formulas so as to incorporate the additional (smooth) factors $\hat{\mathcal{C}}(\xi_w)/\hat{\mathcal{C}}(\xi_{\hat{w}})$ and $\hat{\mathcal{C}}(\xi_w)/\hat{\mathcal{C}}(\xi_{w_0\hat{w}})$, respectively. □

Let $P^{(\mathrm{clas})}_t:\mathcal{H}\mapsto\mathcal{H}$ denote the orthogonal projection onto the finite-dimensional subspace $l^2(\mathcal{P}^+_{\mathrm{clas}}(t))\subset l^2(\mathcal{P}^+)$:

$$(P^{(\mathrm{clas})}_t\phi)_\lambda = \begin{cases}\phi_\lambda & \text{if } \lambda\in\mathcal{P}^+_{\mathrm{clas}}(t),\\ 0 & \text{if } \lambda\in\mathcal{P}^+\setminus\mathcal{P}^+_{\mathrm{clas}}(t).\end{cases} \quad (5.9)$$

It is clear from the definition of the classical wave packet that $P^{(\mathrm{clas})}_t(\phi^{(\mathrm{clas})}(t)) = \phi^{(\mathrm{clas})}(t)$. As a consequence, we get from from Lemma 5.3 upon projection onto $l^2(\mathcal{P}^+_{\mathrm{clas}}(t))$ that

$$\forall\kappa>0: \qquad \|P^{(\mathrm{clas})}_t\phi^{(\infty)}_{\pm}(t)-\phi^{(\mathrm{clas})}(t)\|_{\mathcal{H}} = O(1/|t|^{\kappa}) \quad \text{as } t\to\pm\infty. \quad (5.10)$$

Our final lemma states that the long-time asymptotics of the interacting wave packet $\phi_\pm(t)$ coincides with that of the asymptotic wave packet $\phi^{(\infty)}_{\pm}(t)$.

Lemma 5.4. *For $t\to\pm\infty$, the difference between the interacting wave packet $\phi_\pm(t)$ and the asymptotic wave packet $\phi^{(\infty)}_{\pm}(t)$ tends to zero:*

$$\forall\kappa>0: \qquad \|\phi_\pm(t)-\phi^{(\infty)}_{\pm}(t)\|_{\mathcal{H}} = O(1/|t|^{\kappa}) \quad \text{as } t\to\pm\infty.$$

Proof. From the definitions it is immediate that

$$\phi_{\pm,\lambda}(t)-\phi^{(\infty)}_{\pm,\lambda}(t) = (e^{-it\hat{E}}\hat{\mathcal{S}}_L^{\mp 1/2}\hat{\phi},\overline{\Psi}_\lambda-\overline{\Psi}^{(\infty)}_\lambda)_{\hat{\mathcal{H}}}.$$

Hence

$$\begin{aligned}\|P^{(\mathrm{clas})}_t\left(\phi_\pm(t)-\phi^{(\infty)}_{\pm}(t)\right)\|^2_{\mathcal{H}} &= \sum_{\lambda\in\mathcal{P}^+_{\mathrm{clas}}(t)} |(e^{-it\hat{E}}\hat{\mathcal{S}}_L^{\mp 1/2}\hat{\phi},\overline{\Psi}_\lambda-\overline{\Psi}^{(\infty)}_\lambda)_{\hat{\mathcal{H}}}|^2\\ &\le \|\hat{\phi}\|^2_{\hat{\mathcal{H}}}\sum_{\lambda\in\mathcal{P}^+_{\mathrm{clas}}(t)}\|\Psi_\lambda-\Psi^{(\infty)}_\lambda\|^2_{\hat{\mathcal{H}}}\end{aligned}$$

(by the Cauchy-Schwarz inequality). Now, since $|\mathcal{P}^+_{\mathrm{clas}}(t)| = O(t^N)$ and $m(\lambda) > |t|\varepsilon-1$ for $\lambda\in\mathcal{P}^+_{\mathrm{clas}}(t)$, we conclude from this estimate combined with Theorem 4.1

that $\|P_t^{(\text{clas})}\left(\phi_\pm(t)-\phi_\pm^{(\infty)}(t)\right)\|_{\mathcal{H}}$ converges to zero exponentially fast for $t\to\pm\infty$, so in particular

$$\forall\kappa>0: \qquad \|P_t^{(\text{clas})}\left(\phi_\pm(t)-\phi_\pm^{(\infty)}(t)\right)\|_{\mathcal{H}} = O(1/|t|^\kappa) \quad \text{as } t\to\pm\infty. \tag{5.11}$$

The lemma now follows from the vanishing of the tails $(\text{Id}-P_t^{(\text{clas})})\phi_\pm^{(\infty)}(t)$ and $(\text{Id}-P_t^{(\text{clas})})\phi_\pm(t)$ for $t\to\pm\infty$:

$$\|(\text{Id}-P_t^{(\text{clas})})\phi_\pm^{(\infty)}(t)\|_{\mathcal{H}} = O(1/|t|^\kappa) \quad \text{as } t\to\pm\infty, \tag{5.12a}$$

$$\|(\text{Id}-P_t^{(\text{clas})})\phi_\pm(t)\|_{\mathcal{H}} = O(1/|t|^\kappa) \quad \text{as } t\to\pm\infty. \tag{5.12b}$$

Notice in this connection that the limiting relation in Eq. (5.12a) is immediate from Lemma 5.3 and Eq. (5.10), and that the limiting relation in Eq. (5.12b) follows by comparing the norm equality

$$\|\phi_\pm(t)\|_{\mathcal{H}} = \|\hat{\phi}\|_{\hat{\mathcal{H}}}$$

with the norm estimate for $t\to\pm\infty$

$$\begin{aligned}
\|P_t^{(\text{clas})}\phi_\pm(t)\|_{\mathcal{H}} &\overset{\text{Eq. (5.11)}}{=} \|P_t^{(\text{clas})}\phi_\pm^{(\infty)}(t)\|_{\mathcal{H}} + O(1/|t|^\kappa) \\
&\overset{\text{Eq. (5.10)}}{=} \|\phi_t^{(clas)}\|_{\mathcal{H}} + O(1/|t|^\kappa) \\
&\overset{\text{Lemma 5.2}}{=} \|\phi^{(0)}(t)\|_{\mathcal{H}} + O(1/|t|^\kappa) \\
&= \|\hat{\phi}\|_{\hat{\mathcal{H}}} + O(1/|t|^\kappa),
\end{aligned}$$

which entails that

$$\|(\text{Id}-P_t^{(\text{clas})})\phi_\pm(t)\|_{\mathcal{H}} = \sqrt{\|\phi_\pm(t)\|^2_{\mathcal{H}} - \|P_t^{(\text{clas})}\phi_\pm(t)\|^2_{\mathcal{H}}} = O(1/|t|^{\kappa/2}).$$

□

5.4. **Proof of Proposition 5.1.** After these preparations, the proof of Proposition 5.1 reduces to the telescope

$$\begin{aligned}
&\|\phi_\pm(t)-\phi^{(0)}(t)\|_{\mathcal{H}} \qquad (5.13)\\
&\quad\le \|\phi_\pm(t)-\phi_\pm^{(\infty)}(t)\|_{\mathcal{H}} + \|\phi_\pm^{(\infty)}(t)-\phi^{(clas)}(t)\|_{\mathcal{H}} + \|\phi^{(clas)}(t)-\phi^{(0)}(t)\|_{\mathcal{H}},
\end{aligned}$$

and the application of Lemmas 5.2, 5.3 and 5.4.

6. Application: Scattering of Hyperbolic Lattice Calogero-Moser Models

In this section we specialize our c-functions so as to describe the scattering of the hyperbolic Ruijsenaars-Schneider type lattice Calogero-Moser models associated with the Macdonald polynomials. Initially, viz. in the first three subsections, it will be assumed that our root system $\boldsymbol{R}$ be *reduced* (so $\boldsymbol{R}_0=\boldsymbol{R}_1=\boldsymbol{R}$) except when explicitly stated otherwise. In the fourth subsection, we then indicate how the results extend to the case of a *nonreduced* root system (so $\boldsymbol{R}=BC_N$, $\boldsymbol{R}_0=C_N$ and $\boldsymbol{R}_1=B_N$). We end our study of the lattice Calogero-Moser models in the fifth subsection by providing some illuminating additional details describing what the results boil down to in the simplest situation of a root system of rank *one*.

6.1. **Macdonald Wave Function.** For c-functions of the form

$$\hat{c}_{|\alpha|}(z) = \frac{(q^{g_{|\alpha|}} z; q)_\infty}{(qz; q)_\infty}, \qquad q = e^{-s}, \quad g_{|\alpha|}, s > 0, \tag{6.1}$$

with $(z;q)_\infty \equiv \prod_{n=0}^{\infty}(1 - zq^n)$, the weight function $\hat{\Delta}(\xi)$ (2.5a)–(2.5b) becomes

$$\hat{\Delta}(\xi) = \prod_{\alpha \in \boldsymbol{R}} \frac{(q e^{i\langle \alpha, \xi\rangle}; q)_\infty}{(q^{g_{|\alpha|}} e^{i\langle \alpha, \xi\rangle}; q)_\infty}. \tag{6.2}$$

(The positivity restrictions on the parameters $g_{|\alpha|}$ and s guarantee that the c-function (6.1) meets the technical requirements stated in Section 2.) Our polynomials $P_\lambda(\xi)$, $\lambda \in \mathcal{P}^+$ now amount to the orthonormalized Macdonald polynomials [M2, M3, M4]

$$P_\lambda(\xi) = \frac{1}{\mathcal{N}_0^{1/2}} \Delta^{1/2}(\lambda) \mathbf{P}_\lambda(\xi), \tag{6.3a}$$

where

$$\Delta(\lambda) = \frac{\mathcal{C}^+(\rho_g)\mathcal{C}^-(\rho_g)}{\mathcal{C}^+(\rho_g+\lambda)\mathcal{C}^-(\rho_g+\lambda)}, \qquad \mathcal{N}_0 = \frac{\mathcal{C}^-(\rho_g)}{\mathcal{C}^+(\rho_g)}, \tag{6.3b}$$

with $\rho_g \equiv \frac{1}{2}\sum_{\alpha\in\boldsymbol{R}^+} g_{|\alpha|}\,\alpha$ and

$$\mathcal{C}^\pm(\mathbf{x}) = \prod_{\alpha\in\boldsymbol{R}^+} c^\pm_{|\alpha|}(\langle \mathbf{x}, \alpha^\vee\rangle), \tag{6.3c}$$

$$c^+_{|\alpha|}(x) = q^{g_{|\alpha|}x/2} \frac{(q^{g_{|\alpha|}+x}; q)_\infty}{(q^x; q)_\infty}, \tag{6.3d}$$

$$c^-_{|\alpha|}(x) = q^{g_{|\alpha|}x/2} \frac{(q^{1+x}; q)_\infty}{(q^{1-g_{|\alpha|}+x}; q)_\infty}. \tag{6.3e}$$

Here $\mathbf{P}_\lambda(\xi)$ denotes the Macdonald polynomial

$$\mathbf{P}_\lambda(\xi) = c_\lambda p_\lambda(\xi), \qquad c_\lambda = \frac{\mathcal{C}^+(\rho_g+\lambda)}{\mathcal{C}^+(\rho_g)}, \tag{6.4a}$$

where

$$p_\lambda(\xi) = m_\lambda(\xi) + \sum_{\mu\in\mathcal{P}^+,\, \mu\prec\lambda} c_{\lambda\mu} m_\mu(\xi), \tag{6.4b}$$

with coefficients $c_{\lambda\mu} \in \mathbb{C}$ such that

$$(p_\lambda, m_\mu)_{\hat{\Delta}} = 0 \quad \text{for } \mu \prec \lambda. \tag{6.4c}$$

For the Macdonald weight function $\hat{\Delta}(\xi)$ (6.2), the coefficients $c_{\lambda\mu}$ turn out to vanish when λ and μ are not comparable in the dominance ordering [M3]. In other words, in this case one may take the ordering $\succeq$ to be equal to the dominance order $\geqslant$ (2.8) without restricting generality. Explicit formulas for the expansion coefficients $c_{\lambda\mu}$ when $\mathcal{P} \neq \mathcal{Q}$ (i.e. excluding the root systems E_8, F_4 and G_2) can be found in Ref. [DLM].

We thus arrive at the following formula for the wave function $\Psi_\lambda(\xi)$ (3.2) in terms of Macdonald polynomials.

Proposition 6.1 (Macdonald Wave Function). *For $\boldsymbol{R}$ reduced and c-functions given by $\hat{c}_{|\alpha|}(z)$ (6.1), the wave function $\Psi_\lambda(\xi)$ (3.2) reads explicitly*

$$\Psi_\lambda(\xi) = \frac{1}{\mathcal{N}_0^{1/2}} \Delta^{1/2}(\lambda) \hat{\Delta}^{1/2}(\xi) \delta(\xi) \mathbf{P}_\lambda(\xi),$$

$\boldsymbol{R}$	minuscule	quasi-minuscule
A_N :	$\omega_1, \ldots, \omega_N$	$\omega_1 + \omega_N$,
B_N :	ω_N	ω_1,
C_N :	ω_1	ω_2,
D_N :	$\omega_1, \omega_{N-1}, \omega_N$	ω_2,
E_6 :	ω_1, ω_6	ω_2,
E_7 :	ω_7	ω_1,
E_8 :		ω_8,
F_4 :		ω_4,
G_2 :		ω_1,
BC_N :		ω_1.

Table 1. Minuscule and Quasi-Minuscule Weights.

with $\mathbf{P}_\lambda(\xi)$ *denoting the Macdonald polynomial characterized by Eqs.* (6.4a)–(6.4c).

6.2. **Macdonald-Ruijsenaars Laplacian.** Let us recall that a nonzero weight $\pi \in \mathcal{P}^+$ is called *minuscule* if $\langle \pi, \alpha^\vee \rangle \leq 1$ for all $\alpha \in \boldsymbol{R}^+$ and that it is called *quasi-minuscule* if $\langle \pi, \alpha^\vee \rangle \leq 1$ for all $\alpha \in \boldsymbol{R}^+ \setminus \{\pi\}$ (and it is not minuscule). The number of minuscule weights is equal to the index of $\mathcal{Q}$ in $\mathcal{P}$ minus 1 (so for E_8, F_4 and G_2 there are none). As regards to the quasi-minuscule weights: there is always just *one* and it is given by $\pi = \alpha_0$, where $\alpha_0^\vee$ is the maximal root of the dual root system $\boldsymbol{R}^\vee \equiv \{\alpha^\vee \mid \alpha \in \boldsymbol{R}\}$. For the readers convenience, we have included a list of the (quasi-)minuscule weights for each root system in Table 1 (where we have adopted the standard numbering of the fundamental weights in accordance with Refs. [B, Hu]).

To a (quasi-)minuscule weight π we associate the multiplication operator $\hat{E}_\pi : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$ given by

$$\hat{E}_\pi(\xi) = \sum_{\nu \in W(\pi) \cup W(-\pi)} \exp(i\langle \nu, \xi \rangle). \tag{6.5}$$

The *Macdonald-Ruijsenaars Laplacian* is now defined as the pullback $L_\pi : \mathcal{H} \mapsto \mathcal{H}$ of $\hat{E}_\pi$ with respect to the Fourier transform $\mathcal{F}$

$$L_\pi = \mathcal{F}^{-1} \circ \hat{E}_\pi \circ \mathcal{F}. \tag{6.6}$$

By Proposition 3.2, the Macdonald-Ruijsenaars Laplacian $L_\pi \in \mathbb{R}[L_1, \ldots, L_N]$ constitutes a difference operator in $\mathcal{H}$. The following proposition provides its explicit action on lattice functions over the dominant cone $\mathcal{P}^+$.

Proposition 6.2 (Macdonald-Ruijsenaars Laplacian)**.** *For* $\boldsymbol{R}$ *reduced and* π *(quasi-)minuscule, the action of the Macdonald-Ruijsenaars Laplacian* L_π *on a (square-summable) lattice function* $\phi : \mathcal{P}^+ \to \mathbb{C}$ *is given by*

$$L_\pi \phi_\lambda = E_\pi(\rho_g^\vee)\, \phi_\lambda + \sum_{\substack{\nu \in W(\pi) \cup W(-\pi) \\ \lambda + \nu \in \mathcal{P}^+}} \Big(V_\nu^{1/2}(\rho_g + \lambda) V_{-\nu}^{1/2}(\rho_g + \lambda + \nu) \phi_{\lambda+\nu} - V_\nu(\rho_g + \lambda)\phi_\lambda \Big),$$

where

$$\begin{aligned}
V_\nu(\mathbf{x}) &= \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha^\vee\rangle>0}} \frac{(g_{|\alpha|}+\langle\mathbf{x},\alpha^\vee\rangle:\sinh_s)_{\langle\nu,\alpha^\vee\rangle}}{(\langle\mathbf{x},\alpha^\vee\rangle:\sinh_s)_{\langle\nu,\alpha^\vee\rangle}}\\
&= \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha^\vee\rangle=1}} \frac{\sinh\frac{s}{2}(g_{|\alpha|}+\langle\mathbf{x},\alpha^\vee\rangle)}{\sinh\frac{s}{2}(\langle\mathbf{x},\alpha^\vee\rangle)}\times\\
&\quad \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha^\vee\rangle=2}} \frac{\sinh\frac{s}{2}(g_{|\alpha|}+\langle\mathbf{x},\alpha^\vee\rangle)}{\sinh\frac{s}{2}(\langle\mathbf{x},\alpha^\vee\rangle)}\frac{\sinh\frac{s}{2}(1+g_{|\alpha|}+\langle\mathbf{x},\alpha^\vee\rangle)}{\sinh\frac{s}{2}(1+\langle\mathbf{x},\alpha^\vee\rangle)},\\
E_\pi(\mathbf{x}) &= \sum_{\nu\in W(\pi)\cup W(-\pi)} \exp(s\langle\nu,\mathbf{x}\rangle),
\end{aligned}$$

with $(z:\sinh_s)_m\equiv\prod_{\ell=0}^{m-1}\sinh\frac{s}{2}(z+\ell)$ *and* $\rho_g^\vee\equiv\frac{1}{2}\sum_{\alpha\in\boldsymbol{R}^+}g_{|\alpha|}\alpha^\vee$.

Proof. It is a straightforward consequence of the definitions that the c-functions satisfy the difference equations

$$\frac{c^+_{|\alpha|}(x+1)}{c^+_{|\alpha|}(x)}=\frac{\sinh(\frac{sx}{2})}{\sinh\frac{s}{2}(g_{|\alpha|}+x)},\qquad \frac{c^-_{|\alpha|}(x+1)}{c^-_{|\alpha|}(x)}=\frac{\sinh\frac{s}{2}(1+x-g_{|\alpha|})}{\sinh\frac{s}{2}(1+x)}.$$

With the aid of these difference equations it is not difficult to verify the fundamental functional relation

$$\Delta(\mathbf{x}+\nu)V_{-\nu}(\rho_g+\mathbf{x}+\nu)=\Delta(\mathbf{x})V_\nu(\rho_g+\mathbf{x}),\qquad \nu\in W(\pi). \tag{6.7}$$

From the recurrence relation for the Macdonald polynomials exhibited in Eq. (A.6a) of the Appendix, it is now readily inferred—upon invoking the functional relation (6.7) specialized to $\mathbf{x}=\lambda$ with λ and $\lambda+\nu$ dominant—that the Macdonald wave function $\Psi_\lambda(\xi)$ (6.3c)–(6.3e) satisfies the eigenvalue equation

$$\sum_{\substack{\nu\in W(\pi)\\ \lambda+\nu\in\mathcal{P}^+}}\Big(V_\nu^{1/2}(\rho_g+\lambda)V_{-\nu}^{1/2}(\rho_g+\lambda+\nu)\Psi_{\lambda+\nu}(\xi)-V_\nu(\rho_g+\lambda)\Psi_\lambda(\xi)\Big)$$
$$=\sum_{\nu\in W(\pi)}\big(e^{i\langle\nu,\xi\rangle}-q^{\langle\nu\rho_g^\vee\rangle}\big)\Psi_\lambda(\xi).$$

Combining with the corresponding eigenvalue equation in which π is replaced by $-w_0(\pi)$ (where w_0 is the longest element of the Weyl group), leads us to the eigenvalue equation for L_π:

$$\sum_{\substack{\nu\in W(\pi)\cup W(-\pi)\\ \lambda+\nu\in\mathcal{P}^+}}\big(V_\nu^{1/2}(\rho_g+\lambda)V_{-\nu}^{1/2}(\rho_g+\lambda+\nu)\Psi_{\lambda+\nu}(\xi)-V_\nu(\rho_g+\lambda)\Psi_\lambda(\xi)\Big)$$
$$=\big(\hat{E}_\pi(\xi)-E_\pi(\rho_g^\vee)\big)\Psi_\lambda(\xi).$$

The proposition now follows from the completeness of the Macdonald wave functions $\Psi_\lambda(\xi)$, $\xi\in\mathbf{A}$ in the Hilbert space $\mathcal{H}$. $\square$

When π is minuscule we have that

$$V_\nu(\mathbf{x}) = \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha^\vee\rangle=1}} \frac{\sinh\frac{s}{2}(g_{|\alpha|}+\langle\mathbf{x},\alpha^\vee\rangle)}{\sinh\frac{s}{2}(\langle\mathbf{x},\alpha^\vee\rangle)}, \tag{6.8a}$$

and that

$$\sum_{\nu\in W(\pi)\cup W(-\pi)} V_\nu(\mathbf{x}) = E_\pi(\rho_g^\vee) \tag{6.8b}$$

(by the Macdonald identity in Eq. (A.5) of the Appendix). As a consequence, the action of the Macdonald-Ruijsenaars Laplacian in Proposition 6.2 simplifies in this situation to

$$L_\pi\phi_\lambda = \sum_{\substack{\nu\in W(\pi)\cup W(-\pi)\\ \lambda+\nu\in\mathcal{P}^+}} V_\nu^{1/2}(\rho_g+\lambda)V_{-\nu}^{1/2}(\rho_g+\lambda+\nu)\phi_{\lambda+\nu}. \tag{6.8c}$$

For the root system A_N, all fundamental weights are minuscule (cf. Table 1). Hence, in this special case the discrete Laplacians $L_1,\ldots,L_N$ of Section 3 are given by the Macdonald-Ruijsenaars Laplacians L_π (6.8c) with $\pi=\omega_j$, $j=1,\ldots,N$. The operators in question correspond to the commuting quantum integrals of the hyperbolic relativistic lattice Calogero-Moser model due to Ruijsenaars [R1, R3]. For the other root systems, only a small part of the discrete Laplacians $L_1,\ldots,L_N$ can be made explicit by means of the Macdonald-Ruijsenaars Laplacian of Proposition 6.2 and Table 1. In principle, the higher-order commuting Laplacians may be constructed with the aid of the corresponding Dunkl-Cherednik difference-reflection operators [C1, C2], however, at present explicit formulas for a set of generators for the algebra of commuting Laplacians $\mathbb{R}[L_1,\ldots,L_N]$ are available only in the case of the *classical* root systems [D2, S].

It is of course a consequence of our construction that the algebra of Laplacians $\mathbb{R}[L_1,\ldots,L_N]$ consists of bounded self-adjoint operators in the Hilbert space $\mathcal{H}$. For the Macdonald-Ruijsenaars Laplacian L_π, one can also check this fact independently directly from the explicit action in Propostion 6.2.

Note. If π is quasi-minuscule then $-\pi\in W(\pi)$ (as π is a root). Thus, in this case $W(-\pi)=W(\pi)$. The same simplification also occurs for π not necessarily quasi-minuscule when $-\mathbf{1}\in W$ (i.e. when the longest Weyl-group element w_0 equals $-\mathbf{1}$). This *is* the case for the root systems B_N, C_N, D_N ($N\geq 4$, even), E_7, E_8, F_4, G_2 and BC_N, but it is *not* the case for the root systems A_N ($N\geq 2$), D_N ($N\geq 3$, odd) and E_6.

6.3. **Scattering Matrix.** When $g_{|\alpha|}\to 1$, $\forall\alpha\in\boldsymbol{R}$, the Macdonald c-functions $\hat{c}_{|\alpha|}(z)$ (6.1) specialize to the unit c-function. The Macdonald-Ruijsenaars Laplacian L_π of Proposition 6.2 reduces in this limit to the free Laplacian $L_\pi^{(0)}=(\mathcal{F}^{(0)})^{-1}\circ\hat{E}_\pi\circ\mathcal{F}^{(0)}$ given by

$$L_\pi^{(0)}\phi_\lambda = \sum_{\nu\in W(\pi)\cup W(-\pi)} \phi_{\lambda+\nu} \tag{6.9}$$

with boundary conditions as stipulated in Proposition 3.3. The following proposition provides a somewhat more explicit characterization of these boundary conditions (in the case of π (quasi-)minuscule).

Proposition 6.3 (Action of the Free Laplacian)**.** *For π (quasi-)minuscule, the action of the free Laplacian $L_\pi^{(0)}$ is of the form*

$$L_\pi^{(0)}\phi_\lambda = -n_\pi(\lambda)\phi_\lambda + \sum_{\substack{\nu\in W(\pi)\cup W(-\pi)\\ \lambda+\nu\in\mathcal{P}^+}} \phi_{\lambda+\nu},$$

with $n_\pi(\lambda)=0$ if π is minuscule, and with $n_\pi(\lambda)$ denoting the number of short simple roots α_j perpendicular to λ if π is quasi-minuscule (where, by convention, all roots qualify as short if $\boldsymbol{R}$ is simply laced).

Proof. Starting point is the action of $L_\pi^{(0)}$ in Eq. (6.9) with boundary conditions as detailed in Proposition 3.3. If $\lambda+\nu\notin\mathcal{P}^+$, then there exists a simple root α_j such that $\langle\lambda+\nu,\alpha_j^\vee\rangle<0$. Hence, since $\lambda\in\mathcal{P}^+$ and $\nu\in W(\pi)\cup W(-\pi)$ with π (quasi-)minuscule, it follows that we are in either one of the following three situations:

(i) $\langle\lambda,\alpha_j^\vee\rangle=0$ and $\langle\nu,\alpha_j^\vee\rangle=-1$,
(ii) $\langle\lambda,\alpha_j^\vee\rangle=0$ and $\langle\nu,\alpha_j^\vee\rangle=-2$,
(iii) $\langle\lambda,\alpha_j^\vee\rangle=1$ and $\langle\nu,\alpha_j^\vee\rangle=-2$.

It is not difficult to see that in the first and last situation the weight $\rho+\lambda+\nu$ is stabilized by the simple reflection r_{α_j}. Indeed, we get

$$r_{\alpha_j}(\rho+\lambda+\nu)=\rho+\lambda+\nu-\langle\rho+\lambda+\nu,\alpha_j^\vee\rangle\,\alpha_j=\rho+\lambda+\nu$$

(where we exploited the fact that $\langle\rho,\alpha^\vee\rangle=1$ for α simple). It thus follows that in these two cases the stabilizer of $\rho+\lambda+\nu$ is nontrivial, whence the corresponding term $\phi_{\lambda+\nu}$ in Eq. (6.9) vanishes by the boundary condition in Proposition 3.3. The second situation occurs only when π is quasi-minuscule. Clearly we must then have that $\nu=-\alpha_j$, whence $\langle\rho+\lambda+\nu,\alpha_j^\vee\rangle=-1$ and $\langle\rho+\lambda+\nu,\alpha_k^\vee\rangle= 1+\langle\lambda,\alpha_k^\vee\rangle-\langle\alpha_j,\alpha_k^\vee\rangle>0$ for $k\neq j$ (where we exploited the fact that $\langle\alpha,\beta^\vee\rangle\leq 0$ for α,β simple and distinct). It thus follows that the weight $\rho+\lambda+\nu$ is regular and that the Weyl permutation $w_{\rho+\lambda+\nu}$ taking it to the dominant cone is given by the simple reflection r_{α_j}. Indeed, we now get

$$r_{\alpha_j}(\rho+\lambda+\nu)=\rho+\lambda+\nu-\langle\rho+\lambda+\nu,\alpha_j^\vee\rangle\,\alpha_j=\rho+\lambda.$$

Invoking of the boundary condition in Proposition 3.3 then reveals that the corresponding term $\phi_{\lambda+\nu}$ in Eq. (6.9) is equal to $-\phi_\lambda$. Now, every simple root α_j in the Weyl orbit of π for which $\langle\lambda,\alpha_j^\vee\rangle=0$ gives rise to such a contribution $\phi_{\lambda-\alpha_j}=-\phi_\lambda$ in the action on the r.h.s. of Eq. (6.9). Furthermore, since a quasi-minuscule weight π is a short root of $\boldsymbol{R}$ (as $\pi=\alpha_0$ with $\alpha_0^\vee$ denoting the maximal root of $\boldsymbol{R}^\vee$, whence $\alpha_0^\vee$ is long and α_0 is short), it is clear that the nonzero contributions in question occur precisely at all simple short roots perpendicular to λ. □

Our main application of the scattering formalism in Section 4 is the following explicit formula for the scattering and wave operators for the lattice Calogero-Moser system, relating the long-time asymptotics of the dynamics of the Macdonald-Ruijsenaars Laplacian L_π to that of the free Laplacian $L_\pi^{(0)}$.

Theorem 6.4 (Lattice Calogero-Moser Scattering for $\boldsymbol{R}$ Reduced)**.** *The wave operators $\Omega_\pm = s-\lim_{t\to\pm\infty} e^{itL_\pi}e^{-itL_\pi^{(0)}}$ and the scattering operator $\mathcal{S}_{L_\pi}=\Omega_+^{-1}\Omega_-$ for the Macdonald-Ruijsenaars Laplacian L_π in relation to the free Laplacian $L_\pi^{(0)}$*

are of the form stated in Theorem 4.2 and Corollary 4.3, with a unitary scattering matrix $\hat{S}_{L_\pi}(\xi)$ *given by Eqs.* (4.9), (4.6d) *and*

$$\hat{s}_{|\alpha|}(\langle\alpha,\xi\rangle) = \frac{(qe^{i\langle\alpha,\xi\rangle};q)_\infty}{(q^{g_{|\alpha|}}e^{i\langle\alpha,\xi\rangle};q)_\infty}\frac{(q^{g_{|\alpha|}}e^{-i\langle\alpha,\xi\rangle};q)_\infty}{(qe^{-i\langle\alpha,\xi\rangle};q)_\infty}.$$

For the type A root systems Theorem 6.4 is due to Ruijsenaars [R4]. The scattering of the corresponding classical-mechanical system was analyzed previously in Ref. [R2].

Note. The asymptotics of the Macdonald wave function $\Psi_\lambda(\xi)$ in Proposition 6.1 is governed by Theorem 4.1 (and Eqs. (4.6a)–(4.6e)) with a scattering matrix taken from Theorem 6.4.

6.4. **Extension to Nonreduced Root Systems.** We will now indicate how the results of Subsections 6.1–6.3 should be adapted so as to include the case of a *nonreduced* root system (viz. $\boldsymbol{R} = BC_N$, $\boldsymbol{R}_0 = C_N$, $\boldsymbol{R}_1 = B_N$ and W amounts to the hyperoctahedral group $S_N \ltimes \mathbb{Z}_2^N$). In short, the bottom line is that all results carry over to the case of nonreduced root systems upon passing from the Macdonald polynomials to the Macdonald-Koornwinder multivariate Askey-Wilson polynomials [K, D2, S]. More specifically, by picking c-functions $\hat{c}_{|\alpha|}(z)$, $\alpha\in\boldsymbol{R}_1$ of the form

$$\hat{c}_{|\alpha|}(z) = \begin{cases} \dfrac{(q^{\hat{g}}z;q)_\infty}{(qz;q)_\infty} & \text{for } \alpha \text{ long,} \\[2ex] \dfrac{(q^{\hat{g}_0}z,-q^{\hat{g}_1}z,q^{\hat{g}_2+1/2}z,-q^{\hat{g}_3+1/2}z;q)_\infty}{(qz^2;q)_\infty} & \text{for } \alpha \text{ short,} \end{cases} \tag{6.10}$$

where $q=e^{-s}$ and $s,\hat{g},\hat{g}_0,\ldots,\hat{g}_3>0$ (and with $(z_1,\ldots,z_k;q)_\infty \equiv (z_1;q)_\infty\cdots(z_k;q)_\infty$), we end up with a weight function $\hat{\Delta}(\xi)$ (2.5a)–(2.5b) given by

$$\begin{aligned}\hat{\Delta}(\xi) = &\prod_{\substack{\alpha\in\boldsymbol{R}_1\\ \alpha \text{ long}}}\frac{(qe^{i\langle\alpha,\xi\rangle};q)_\infty}{(q^{\hat{g}}e^{i\langle\alpha,\xi\rangle};q)_\infty} \\ &\times\prod_{\substack{\alpha\in\boldsymbol{R}_1\\ \alpha \text{ short}}}\frac{(qe^{2i\langle\alpha,\xi\rangle};q)_\infty}{(q^{\hat{g}_0}e^{i\langle\alpha,\xi\rangle},-q^{\hat{g}_1}e^{i\langle\alpha,\xi\rangle},q^{\hat{g}_2+1/2}e^{i\langle\alpha,\xi\rangle},-q^{\hat{g}_3+1/2}e^{i\langle\alpha,\xi\rangle};q)_\infty}.\end{aligned} \tag{6.11}$$

The polynomials $P_\lambda(\xi)$, $\lambda\in\mathcal{P}^+$ amount in this case to orthonormalized Macdonald-Koornwinder polynomials [K, D2, S]. The polynomials in question are again of the form in Eqs. (6.3a), (6.3b) and Eqs. (6.4a)–(6.4c), but now with

$$\mathcal{C}^\pm(\mathbf{x}) = \prod_{\alpha\in\boldsymbol{R}_1^+} c^\pm_{|\alpha|}(\langle\mathbf{x},\alpha\rangle), \tag{6.12a}$$

$$c^+_{|\alpha|}(x) = \begin{cases} q^{gx/2}\dfrac{(q^{g+x};q)_\infty}{(q^x;q)_\infty} & \text{for } \alpha \text{ long,} \\[2ex] q^{(g_0+g_1+g_2+g_3)x/2}\times \\ \dfrac{(q^{g_0+x},-q^{g_1+x},q^{g_2+1/2+x},-q^{g_3+1/2+x};q)_\infty}{(q^{2x};q)_\infty} & \text{for } \alpha \text{ short,} \end{cases} \tag{6.12b}$$

$$c^-_{|\alpha|}(x) = \tag{6.12c}$$
$$\begin{cases} q^{gx/2}\dfrac{(q^{1+x};q)_\infty}{(q^{1-g+x};q)_\infty} & \text{for } \alpha \text{ long}, \\ q^{(g_0+g_1+g_2+g_3)x/2}\times & \\ \dfrac{(q^{1+2x};q)_\infty}{(q^{1-g_0+x},-q^{1-g_1+x},q^{1/2-g_2+x},-q^{1/2-g_3+x};q)_\infty} & \text{for } \alpha \text{ short}, \end{cases}$$

where we have distinguished dual parameters g, $g_0,\ldots,g_3$ that are related to the parameters $\hat{g}$, $\hat{g}_0,\ldots,\hat{g}_3$ via the linear relations

$$g=\hat{g}, \qquad \begin{pmatrix} g_0\\ g_1\\ g_2\\ g_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix} \hat{g}_0\\ \hat{g}_1\\ \hat{g}_2\\ \hat{g}_3\end{pmatrix}, \tag{6.13a}$$

and with the vectors ρ_g and $\rho_g^\vee$ now taken to be

$$\rho_g = \frac{g}{2}\sum_{\substack{\alpha\in\boldsymbol{R}_1^+\\ \alpha \text{ long}}}\alpha + g_0\sum_{\substack{\alpha\in\boldsymbol{R}_1^+\\ \alpha \text{ short}}}\alpha, \tag{6.13b}$$

$$\rho_g^\vee = \frac{\hat{g}}{2}\sum_{\substack{\alpha\in\boldsymbol{R}_1^+\\ \alpha \text{ long}}}\alpha + \hat{g}_0\sum_{\substack{\alpha\in\boldsymbol{R}_1^+\\ \alpha \text{ short}}}\alpha. \tag{6.13c}$$

We thus arrive at the following formula for the wave function $\Psi_\lambda(\xi)$ (3.2) in terms of Macdonald-Koornwinder polynomials.

Proposition 6.5 (Macdonald-Koornwinder Wave Function)**.** *For $\boldsymbol{R}$ nonreduced and c-functions given by $\hat{c}_{|\alpha|}(z)$ (6.10), the wave function $\Psi_\lambda(\xi)$ (3.2) reads explicitly*

$$\Psi_\lambda(\xi) = \frac{1}{\mathcal{N}_0^{1/2}}\Delta^{1/2}(\lambda)\hat{\Delta}^{1/2}(\xi)\delta(\xi)\mathbf{P}_\lambda(\xi),$$

with $\mathbf{P}_\lambda(\xi)$ denoting the Macdonald-Koornwinder polynomial characterized by Eqs. (6.4a)–(6.4c).

From the second-order recurrence relation for the Macdonald-Koornwinder polynomials [D2], we now obtain the following formula for the action of the Macdonald-Koornwinder Laplacian $L_\pi = \mathcal{F}^{-1}\circ\hat{E}_\pi\circ\mathcal{F}$, associated to the first fundamental weight $\pi=\omega_1$ (which is a quasi-minuscule weight for $\boldsymbol{R}=BC_N$, cf. Table 1).

Proposition 6.6 (Macdonald-Koornwinder Laplacian)**.** *For $\boldsymbol{R}$ nonreduced and $\pi=\omega_1$, the action of the Macdonald-Koorwinder Laplacian L_π on a (square-summable) lattice function $\phi:\mathcal{P}^+\to\mathbb{C}$ is given by*

$$L_\pi\phi_\lambda = E_\pi(\rho_g^\vee)\phi_\lambda + \sum_{\substack{\nu\in W(\pi)\\ \lambda+\nu\in\mathcal{P}^+}}\Big(V_\nu^{1/2}(\rho_g+\lambda)V_{-\nu}^{1/2}(\rho_g+\lambda+\nu)\phi_{\lambda+\nu} - V_\nu(\rho_g+\lambda)\phi_\lambda\Big),$$

where

$$
\begin{aligned}
V_\nu(\mathbf{x}) \;&=\; \prod_{\substack{\alpha\in \boldsymbol{R}_1\\ \alpha\ long,\ \langle \nu,\alpha\rangle =1}} \frac{\sinh \frac{s}{2}(g+\langle \mathbf{x},\alpha\rangle)}{\sinh \frac{s}{2}(\langle \mathbf{x},\alpha\rangle)}\times \\
&\quad \prod_{\substack{\alpha\in \boldsymbol{R}_1\\ \alpha\ short,\ \langle \nu,\alpha\rangle =1}} \frac{\sinh \frac{s}{2}(g_0+\langle \mathbf{x},\alpha\rangle)}{\sinh \frac{s}{2}(\langle \mathbf{x},\alpha\rangle)}\frac{\cosh \frac{s}{2}(g_1+\langle \mathbf{x},\alpha\rangle)}{\cosh \frac{s}{2}(\langle \mathbf{x},\alpha\rangle)} \\
&\qquad\qquad \times \frac{\sinh \frac{s}{2}(g_2+\frac{1}{2}+\langle \mathbf{x},\alpha\rangle)}{\sinh \frac{s}{2}(\frac{1}{2}+\langle \mathbf{x},\alpha\rangle)}\frac{\cosh \frac{s}{2}(g_3+\frac{1}{2}+\langle \mathbf{x},\alpha\rangle)}{\cosh \frac{s}{2}(\frac{1}{2}+\langle \mathbf{x},\alpha\rangle)}, \\
E_\pi(\mathbf{x}) \;&=\; \sum_{\nu\in W(\pi)} \exp(s\langle \nu,\mathbf{x}\rangle),
\end{aligned}
$$

and with ρ_g *and* $\rho_g^\vee$ *given by Eqs.* (6.13b) *and* (6.13c), *respectively.*

For $\hat g \longrightarrow 1$ and $\hat g_0,\ldots,\hat g_3\longrightarrow 1/2$ (so $g, g_0\longrightarrow 1$ and $g_1,g_2,g_3\longrightarrow 0$), the c-functions $\hat c_{|\alpha|}(z)$ (6.10) tend to 1 (recall in this connection the duplication formula $(z^2;q)_\infty = (z,-z,q^{1/2}z,-q^{1/2}z;q)_\infty$). The Macdonald-Koornwinder Laplacian L_π then reduces to the free Laplacian

$$
L_\pi^{(0)}\phi_\lambda = \sum_{\nu\in W(\pi),\, \lambda+\nu\in\mathcal{P}^+} \phi_{\lambda+\nu}. \tag{6.14}
$$

Application of the scattering formalism of Section 4 now produces the following scattering and wave operators relating the long-time asymptotics of the dynamics of the Macdonald-Koornwinder Laplacian L_π to that of the free Laplacian $L_\pi^{(0)}$ (6.14).

Theorem 6.7 (Lattice Calogero-Moser Scattering for $\boldsymbol{R}$ Nonreduced)**.** *The wave operators* $\Omega_\pm = s{-}\lim_{t\to\pm\infty} e^{itL_\pi}e^{-itL_\pi^{(0)}}$ *and the scattering operator* $\mathcal{S}_{L_\pi}=\Omega_+^{-1}\Omega_-$ *for the Macdonald-Koornwinder Laplacian* L_π *in relation to the free Laplacian* $L_\pi^{(0)}$ (6.14) *are of the form stated in Theorem 4.2 and Corollary 4.3, with a unitary scattering matrix* $\hat{\mathcal{S}}_{L_\pi}$ *given by Eqs.* (4.9), (4.6d) *and*

$$
\hat s_{|\alpha|}(\langle\alpha,\xi\rangle) =
\begin{cases}
\dfrac{(qe^{i\langle\alpha,\xi\rangle};q)_\infty}{(q^{\hat g}e^{i\langle\alpha,\xi\rangle};q)_\infty}\dfrac{(q^{\hat g}e^{-i\langle\alpha,\xi\rangle};q)_\infty}{(qe^{-i\langle\alpha,\xi\rangle};q)_\infty} & \text{for } \alpha \text{ long,}\\[2ex]
\dfrac{(qe^{2i\langle\alpha,\xi\rangle};q)_\infty}{(q^{\hat g_0}e^{i\langle\alpha,\xi\rangle},-q^{\hat g_1}e^{i\langle\alpha,\xi\rangle},q^{\hat g_2+1/2}e^{i\langle\alpha,\xi\rangle},-q^{\hat g_3+1/2}e^{i\langle\alpha,\xi\rangle};q)_\infty}\times & \\
\dfrac{(q^{\hat g_0}e^{-i\langle\alpha,\xi\rangle},-q^{\hat g_1}e^{-i\langle\alpha,\xi\rangle},q^{\hat g_2+1/2}e^{-i\langle\alpha,\xi\rangle},-q^{\hat g_3+1/2}e^{-i\langle\alpha,\xi\rangle};q)_\infty}{(qe^{-2i\langle\alpha,\xi\rangle};q)_\infty} & \text{for } \alpha \text{ short.}
\end{cases}
$$

Note. The asymptotics of the Macdonald-Koornwinder wave function $\Psi_\lambda(\xi)$ in Proposition 6.5 is governed by Theorem 4.1 (and Eqs. (4.6a)–(4.6e)) with scattering matrices taken from Theorem 6.7.

6.5. **Example: The Rank-One Case.** It is quite instructive to exhibit the results of this section in somewhat further detail for simplest case of a root system of rank *one*. We will restrict attention the case of a nonreduced root system (i.e. BC_1), since the reduced case (viz. A_1) can be recovered from it via a specialization

of the parameters (corresponding to a standard reduction from the Askey-Wilson polynomials to the q-ultraspherical polynomials [AW, GS]).

In this situation the Macdonald-Koornwinder wave function takes the explicit basic hypergeometric form

$$\Psi_l(\xi) = \frac{1}{\mathcal{N}_0^{1/2}} \Delta^{1/2}(\ell)\hat{\Delta}^{1/2}(\xi)\delta(\xi)\mathbf{P}_\ell(\xi), \qquad \ell \in \mathbb{N}, \quad \xi \in (0,\pi), \tag{6.15}$$

where

$$\mathcal{N}_0 = \frac{c^-(g_0)}{c^+(g_0)}, \tag{6.16a}$$

$$\Delta(\ell) = \frac{c^+(g_0)c^-(g_0)}{c^+(g_0+\ell)c^-(g_0+\ell)}, \tag{6.16b}$$

$$\hat{\Delta}(\xi) = \frac{1}{\hat{c}(\xi)\hat{c}(-\xi)}, \tag{6.16c}$$

$$\delta(\xi) = 2\sin(\xi), \tag{6.16d}$$

with

$$\hat{c}(\xi) = \frac{(q^{\hat{g}_0}e^{-i\xi}, -q^{\hat{g}_1}e^{-i\xi}, q^{\hat{g}_2+1/2}e^{-i\xi}, -q^{\hat{g}_3+1/2}e^{-i\xi}; q)_\infty}{(qe^{-2i\xi}; q)_\infty}, \tag{6.17a}$$

$$c^+(x) = q^{(g_0+g_1+g_2+g_3)x/2} \times \frac{(q^{g_0+x}, -q^{g_1+x}, q^{g_2+1/2+x}, -q^{g_3+1/2+x}; q)_\infty}{(q^{2x}; q)_\infty}, \tag{6.17b}$$

$$c^-(x) = q^{(g_0+g_1+g_2+g_3)x/2} \times \frac{(q^{1+2x}; q)_\infty}{(q^{1-g_0+x}, -q^{1-g_1+x}, q^{1/2-g_2+x}, -q^{1/2-g_3+x}; q)_\infty}, \tag{6.17c}$$

and with $\mathbf{P}_\ell(\xi)$ denoting the Askey-Wilson polynomial [AW, GS]

$$\mathbf{P}_\ell(\xi) = {}_4\Phi_3\left(\begin{matrix} q^{-\ell}, q^{2g_0+\ell}, q^{\hat{g}_0}e^{i\xi}, q^{\hat{g}_0}e^{-i\xi} \\ -q^{g_0+g_1}, q^{g_0+g_2+1/2}, -q^{g_0+g_3+1/2} \end{matrix} ; q, q \right). \tag{6.18}$$

Here we have employed standard notation from the theory of basic hypergeometric series [GS]

$${}_s\Phi_{s-1}\left(\begin{matrix} a_1, \ldots, a_s \\ b_1, \ldots, b_{s-1} \end{matrix} ; q, z \right) \equiv \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_s; q)_n}{(b_1, \ldots, b_{s-1}; q)_n} \frac{z^n}{(q;q)_n},$$

with $(a;q)_n \equiv \prod_{k=0}^{n-1}(1-aq^k)$ and $(a_1, \ldots, a_s; q)_n \equiv (a_1;q)_n \cdots (a_s;q)_n$.

The asymptotics of the wave function $\Psi_l(\xi)$ for $\ell \longrightarrow \infty$ is given by (cf. also [IW, I])

$$\Psi_l^\infty(\xi) = \hat{s}^{1/2}(\xi)e^{i(\ell+1)\xi} - \hat{s}^{-1/2}(\xi)e^{-i(\ell+1)\xi}, \tag{6.19a}$$

with

$$\hat{s}(\xi) = \frac{\hat{c}(\xi)}{\hat{c}(-\xi)}. \tag{6.19b}$$

The free plane waves $\Psi_\ell^{(0)}(\xi)$ (3.4) boil in this case down to the Fourier sine kernel

$$\Psi_\ell^{(0)}(\xi) = 2\sin(\ell+1)\xi, \qquad \ell \in \mathbb{N}, \quad \xi \in (0,\pi). \tag{6.20}$$

The corresponding Fourier pairings $\mathcal{F} : l^2(\mathbb{N}) \mapsto L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ and $\mathcal{F}^{(0)} : l^2(\mathbb{N}) \mapsto L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ together with their inversion formulas are given by

$$\begin{cases} \hat{\phi}(\xi) = \sum_{\ell\in\mathbb{N}} \phi_\ell \Psi_\ell(\xi), \\ \phi_\ell = \frac{1}{2\pi}\int_0^\pi \hat{\phi}(\xi)\Psi_\ell(\xi)\mathrm{d}\xi, \end{cases} \tag{6.21a}$$

and

$$\begin{cases} \hat{\phi}(\xi) = \sum_{\ell\in\mathbb{N}} \phi_\ell \Psi_\ell^{(0)}(\xi), \\ \phi_\ell = \frac{1}{2\pi}\int_0^\pi \hat{\phi}(\xi)\Psi_\ell^{(0)}(\xi)\mathrm{d}\xi, \end{cases} \tag{6.21b}$$

respectively (where we have omitted the complex conjugations because the relevant kernel functions $\Psi_\ell(\xi)$ and $\Psi_\ell^{(0)}(\xi)$ are real-valued as a consequence of the fact that $-\mathbf{1} \in W \cong \mathbb{Z}_2$).

The Macdonald-Koornwinder Laplacian $L = \mathcal{F}^{-1}\circ\hat{E}\circ\mathcal{F}$ and the free Laplacian $L^{(0)} = (\mathcal{F}^{(0)})^{-1}\circ\hat{E}\circ\mathcal{F}^{(0)}$ associated to the multiplication operator $\hat{E}(\xi) = 2\cos(\xi)$ act on on lattice functions $\phi \in l^2(\mathbb{N})$ respectively as

$$\begin{aligned} L\phi_\ell \;=\;& V^{1/2}(g_0+\ell)V^{1/2}(-g_0-\ell-1)\phi_{\ell+1} + \\ & V^{1/2}(-g_0-\ell)V^{1/2}(g_0+\ell-1)\phi_{\ell-1} + \\ & \big(2\cosh(s\hat{g}_0) - V(g_0+\ell) - (1-\delta_{\ell,0})V(-g_0-\ell)\big)\phi_\ell, \end{aligned} \tag{6.22a}$$

with

$$\begin{aligned} V(x) = \;& \frac{\sinh\frac{s}{2}(g_0+x)}{\sinh\frac{s}{2}(x)}\frac{\cosh\frac{s}{2}(g_1+x)}{\cosh\frac{s}{2}(x)} \\ & \times \frac{\sinh\frac{s}{2}(g_2+\frac{1}{2}+x)}{\sinh\frac{s}{2}(\frac{1}{2}+x)}\frac{\cosh\frac{s}{2}(g_3+\frac{1}{2}+x)}{\cosh\frac{s}{2}(\frac{1}{2}+x)}, \end{aligned} \tag{6.22b}$$

and as

$$L^{(0)}\phi_\ell = \phi_{\ell+1} + \phi_{\ell-1}, \tag{6.23}$$

with the boundary condition $\phi_{-1} = 0$.

The specialization of Theorem 6.7 to the case $N = 1$ now states that the wave operators $\Omega_\pm = s - \lim_{t\to\pm\infty} e^{itL}e^{-itL^{(0)}}$ and the scattering operator $\mathcal{S}_L = \Omega_+^{-1}\Omega_-$ exist in $l^2(\mathbb{N})$, and are moreover of the form $\Omega_\pm = \mathcal{F}^{-1}\circ\hat{\mathcal{S}}^{\mp 1/2}\circ\mathcal{F}^{(0)}$ and $\mathcal{S}_L = (\mathcal{F}^{(0)})^{-1}\circ\hat{\mathcal{S}}_L\circ\mathcal{F}^{(0)}$, respectively, with $\hat{\mathcal{S}}_L$ being a unitary scattering matrix whose multiplicative action on a wave packet $\hat{\phi} \in L^2((0,\pi),(2\pi)^{-1}\mathrm{d}\xi)$ is given by

$$(\hat{\mathcal{S}}_L\hat{\phi})(\xi) = \frac{\hat{c}(-\xi)}{\hat{c}(\xi)}\,\hat{\phi}(\xi) \quad \text{for } 0 < \xi < \pi. \tag{6.24}$$

Appendix A. Properties of the Macdonald Polynomials

This appendix serves to list a number of key properties of the Macdonald polynomials $\mathbf{P}_\lambda(\xi)$, $\lambda \in \mathcal{P}^+$ defined by Eqs. (6.4a)–(6.4c). We used these properties in Section 6 to build the Macdonald wave function and to the determine the explicit action of the Macdonald-Ruijsenaars Laplacian. For proofs of the statements below and for further theory concerning the Macdonald polynomials the reader is referred to the seminal works of Macdonald and Cherednik [M2, M3, M4, C1, C2] (see also [C] for a different approach). Throughout this appendix it is assumed that our

root system $\boldsymbol{R}$ be *reduced*. For the extension of the statements below to the case of *nonreduced* root systems the reader is referred to Refs. [M3, M4, K, D2, Ok, S].

The Macdonald polynomials $\mathbf{P}_\lambda(\xi)$ (6.4a)–(6.4c) are normalized such that they satisfy the *Specialization Formula*

$$\mathbf{P}_\lambda(is\rho_g^\vee) = 1. \tag{A.1}$$

In this normalization the *Orthogonality Relations* read

$$(\mathbf{P}_\lambda, \mathbf{P}_\mu)_{\hat{\mathcal{H}}} = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ \frac{\mathcal{N}_0}{\Delta(\lambda)} & \text{if } \lambda = \mu. \end{cases} \tag{A.2}$$

The specialization formula in Eq. (A.1) amounts to the special case $\mu = 0$ of the more general *Symmetry Relation*

$$\mathbf{P}_\lambda^{R}(is(\rho_g^\vee + \mu)) = \mathbf{P}_\mu^{R^\vee}(is(\rho_g + \lambda)), \tag{A.3}$$

where $\mathbf{P}_\lambda^{R}(\xi)$ and $\mathbf{P}_\mu^{R^\vee}(\xi)$ refer to the Macdonald polynomials associated to the root system $\boldsymbol{R}$ and the the dual root system $\boldsymbol{R}^\vee$, respectively (so λ and μ are dominant weights of $\boldsymbol{R}$ and $\boldsymbol{R}^\vee$, respectively).

For any (quasi-)minuscule weight π of $\boldsymbol{R}^\vee$, we have a corresponding *Macdonald Difference Equation* given by

$$\begin{aligned}\sum_{\nu\in W(\pi)} \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha\rangle>0}} \frac{(isg_{|\alpha|} + \langle\xi,\alpha\rangle : \sin)_{\langle\nu,\alpha\rangle}}{(\langle\xi,\alpha\rangle : \sin)_{\langle\nu,\alpha\rangle}} \bigl(\mathbf{P}_\lambda(\xi + is\nu) - \mathbf{P}_\lambda(\xi)\bigr) \\ = \sum_{\nu\in W(\pi)} \bigl(q^{\langle\nu,\lambda+\rho_g\rangle} - q^{\langle\nu,\rho_g\rangle}\bigr)\mathbf{P}_\lambda(\xi),\end{aligned} \tag{A.4a}$$

where $(z:\sin)_m \equiv \prod_{\ell=0}^{m-1} \sin\frac{1}{2}(z+is\ell)$. If the weight π is minuscule (so $|\langle\pi,\alpha\rangle| \leq 1$, $\forall\alpha\in\boldsymbol{R}$), then this difference equation simplifies to

$$\sum_{\nu\in W(\pi)} \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha\rangle=1}} \frac{\sin\frac{1}{2}(isg_{|\alpha|} + \langle\xi,\alpha\rangle)}{\sin\frac{1}{2}(\langle\xi,\alpha\rangle)} \mathbf{P}_\lambda(\xi + is\nu) = \sum_{\nu\in W(\pi)} q^{\langle\nu,\lambda+\rho_g\rangle}\mathbf{P}_\lambda(\xi), \tag{A.4b}$$

because of the *Macdonald Identity* (for π minuscule)

$$\sum_{\nu\in W(\pi)} \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha\rangle=1}} \frac{\sin\frac{1}{2}(isg_{|\alpha|} + \langle\xi,\alpha\rangle)}{\sin\frac{1}{2}(\langle\xi,\alpha\rangle)} = \sum_{\nu\in W(\pi)} q^{\langle\nu,\rho_g\rangle}. \tag{A.5}$$

Combination of the Macdonald difference equation in Eq. (A.4a) and the symmetry relation in Eq. (A.3) leads to the *Recurrence Relation (or Pieri Formula)*

$$\begin{aligned}\sum_{\nu\in W(\pi)} \bigl(e^{i\langle\nu,\xi\rangle} - q^{\langle\nu,\rho_g^\vee\rangle}\bigr)\mathbf{P}_\lambda(\xi) = \\ \sum_{\substack{\nu\in W(\pi)\\ \lambda+\nu\in\mathcal{P}^+}} \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha^\vee\rangle>0}} \frac{(g_{|\alpha|} + \langle\rho_g+\lambda,\alpha^\vee\rangle : \sinh_s)_{\langle\nu,\alpha^\vee\rangle}}{(\langle\rho_g+\lambda,\alpha^\vee\rangle : \sinh_s)_{\langle\nu,\alpha^\vee\rangle}} \bigl(\mathbf{P}_{\lambda+\nu}(\xi) - \mathbf{P}_\lambda(\xi)\bigr),\end{aligned} \tag{A.6a}$$

where π is now a (quasi-)minuscule weight of $\boldsymbol{R}$ (and $(z:\sinh_s)_m \equiv \prod_{\ell=0}^{m-1}\sinh(\frac{s}{2}(z+\ell)))$. In the minuscule case this recurrence relation reduces to

$$\sum_{\nu\in W(\pi)} e^{i\langle\nu,\xi\rangle}\mathbf{P}_\lambda(\xi) = \sum_{\substack{\nu\in W(\pi)\\ \lambda+\nu\in\mathcal{P}^+}} \prod_{\substack{\alpha\in\boldsymbol{R}\\ \langle\nu,\alpha^\vee\rangle=1}} \frac{\sinh\frac{s}{2}(g_{|\alpha|}+\langle\rho_g+\lambda,\alpha^\vee\rangle)}{\sinh\frac{s}{2}(\langle\rho_g+\lambda,\alpha^\vee\rangle)}\mathbf{P}_{\lambda+\nu}(\xi). \quad \text{(A.6b)}$$

Appendix B. Index of Notations

This Appendix provides a list of notations ordered according to the sections in which they were first introduced.

Section 2.1: $\mathbf{E}$, $\langle\cdot,\cdot\rangle$, $\boldsymbol{R}$, $\boldsymbol{R}^+$, $\mathcal{Q}$, $\mathcal{Q}^+$, $\mathcal{P}$, $\mathcal{P}^+$, $\alpha^\vee$, $\mathbf{A}$, $m_\lambda(\xi)$, ξ_w, W, W_λ, $|W_\lambda|$.

Section 2.2: $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, $\hat\Delta(\xi)$, $\hat{\mathcal{C}}(\xi)$, $\hat c_{|\alpha|}(z)$.

Section 2.3: $(\cdot,\cdot)_{\hat\Delta}$, $\mathrm{Vol}(\mathbf{A})$, $\delta(\xi)$, $\succeq$, $\geqslant$, $P_\lambda(\xi)$, $a_{\lambda\mu}$.

Section 2.4: $\chi_\lambda(\xi)$, $(-1)^w$, ρ, w_μ.

Section 3.1: $\mathcal{H}$, $(\cdot,\cdot)_{\mathcal{H}}$, $\hat{\mathcal{H}}$, $(\cdot,\cdot)_{\hat{\mathcal{H}}}$, $\Psi_\lambda(\xi)$, $\mathcal{F}$, $\Psi^{(0)}_\lambda(\xi)$, $\mathcal{F}^{(0)}$.

Section 3.2: ω_r, $\hat E_r(\xi)$, $W(\cdot)$, L_r, $\sigma(L_r)$, w_0.

Section 3.3: $a_{\lambda\mu;r}$, $\mathcal{P}^+_{\lambda;r}$, $L^{(0)}_r$.

Section 4.1: $\mathbf{C}^+$, $m(\lambda)$, $P^\infty_\lambda(\xi)$, $\|\cdot\|_{\hat\Delta}$, $P^{m(\lambda)}_\lambda(\xi)$, $\Psi^\infty_\lambda(\xi)$, $\hat S_w(\xi)$, $\hat s_{|\alpha|}(\langle\alpha,\xi\rangle)$, $\|\cdot\|_{\hat{\mathcal{H}}}$.

Section 4.2: $\hat E(\xi)$, L, $L^{(0)}$, $\sigma(L)$, $\mathbf{A}_{\mathrm{reg}}$, $\hat w_\xi$, $\hat{\mathcal{S}}_L$, $\|\cdot\|_{\mathcal{H}}$, $\Omega_\pm$, $\mathcal{S}_L$.

Section 5.1: $\phi^{(0)}(t)$, $\phi_\pm(t)$.

Section 5.2: $\hat w$, $\mathbf{V}_{\mathrm{clas}}$, $\mathcal{P}^+_{\mathrm{clas}}(t)$, $\phi^{(\mathrm{clas})}(t)$, $\hat W$.

Section 5.3: $\phi^{(\infty)}_\pm$, $P^{(\mathrm{clas})}_t$.

Section 6.1: $g_{|\alpha|}$, $(z;q)_\infty$, $\Delta(\lambda)$, $\mathcal{N}_0$, $\mathcal{C}^\pm(\mathbf{x})$, ρ_g, $c^\pm_{|\alpha|}(x)$, $\mathbf{P}_\lambda(\xi)$.

Section 6.2: π, $\alpha_0^\vee$, $\boldsymbol{R}^\vee$, $\hat E_\pi(\xi)$, L_π, $V_\nu(\mathbf{x})$, $E_\pi(\mathbf{x})$, $(z:\sinh_s)_m$, $\rho_g^\vee$.

Section 6.3: $L^{(0)}_\pi$, $n_\pi(\lambda)$, α_j, r_{α_j}, $\mathcal{S}_{L_\pi}$, $\hat{\mathcal{S}}_{L_\pi}$.

Section 6.4: $\hat g$, $\hat g_r$, $(z_1,\ldots,z_k;q)_\infty$, g, g_r.

Section 6.5: ${}_{s+1}\Phi_s$, $(a;q)_n$, $(a_1,\ldots,a_s;q)_n$.

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